n-Player Stochastic Games with Additive Transitions

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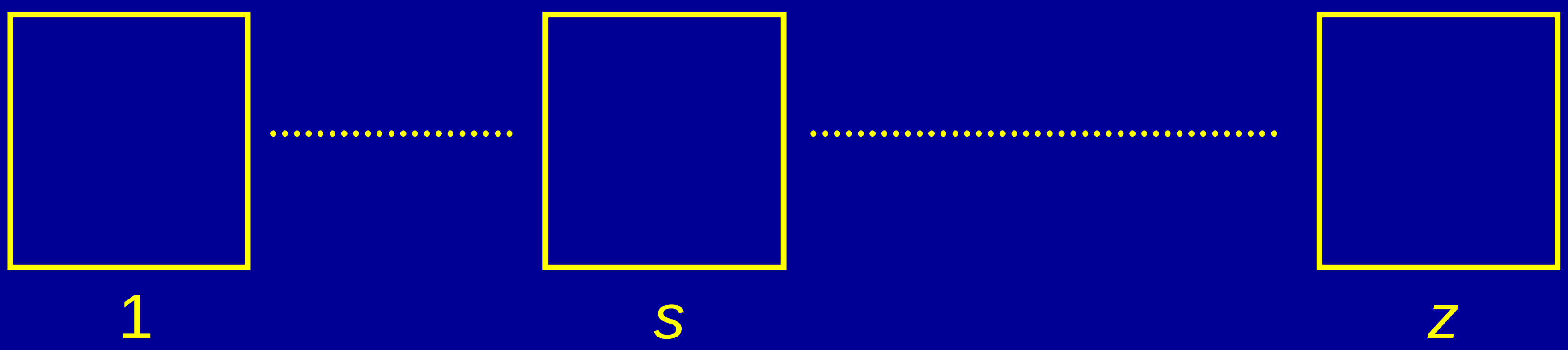
European Journal of Operational Research 179 (2007) 483–497

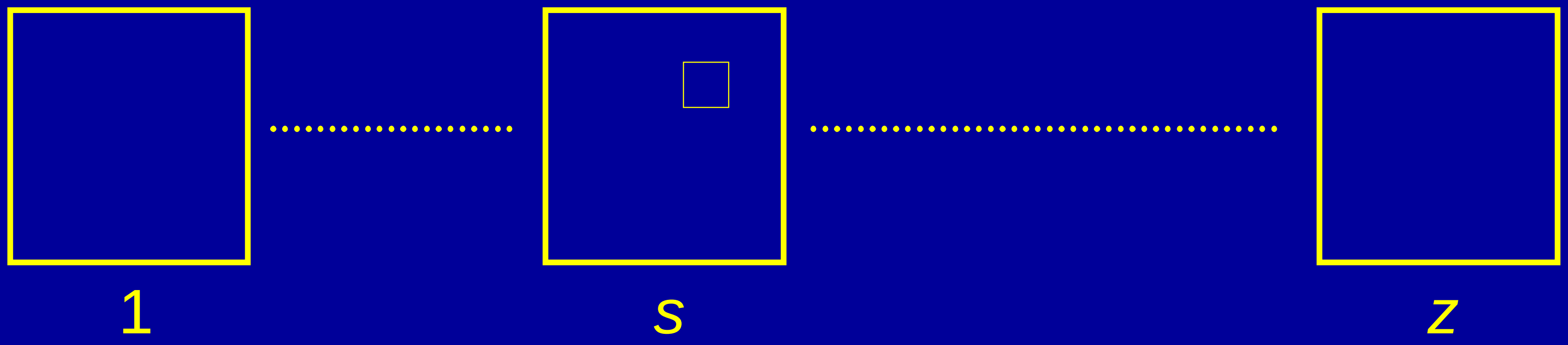
Outline

- Model
- Brief History of Stochastic Games
- Additive Transitions
- Examples

Finite Stochastic Game

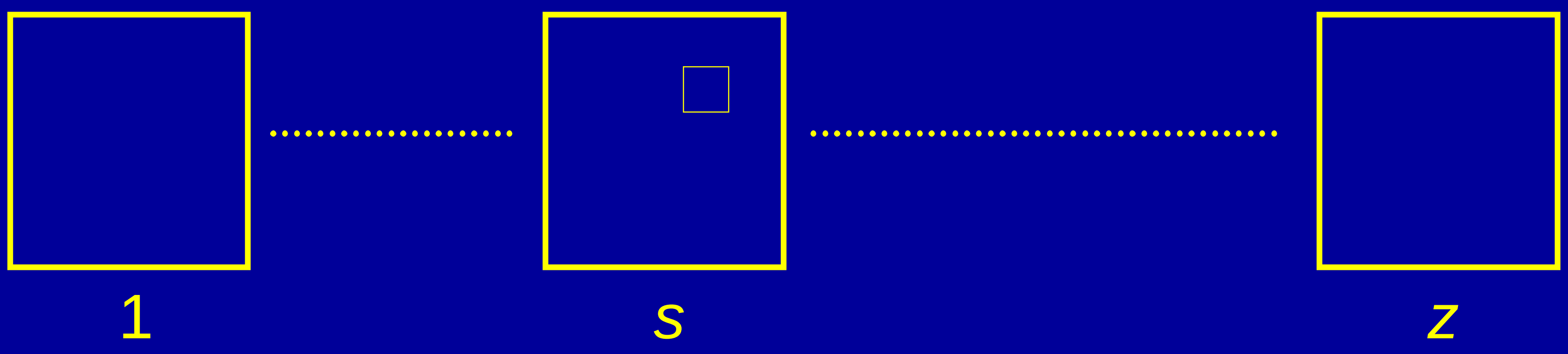


Finite Stochastic Game



$a_s = (a^1_s\,, a^2_s\,, \ldots\,, a^n_s)$ joint action

Finite Stochastic Game

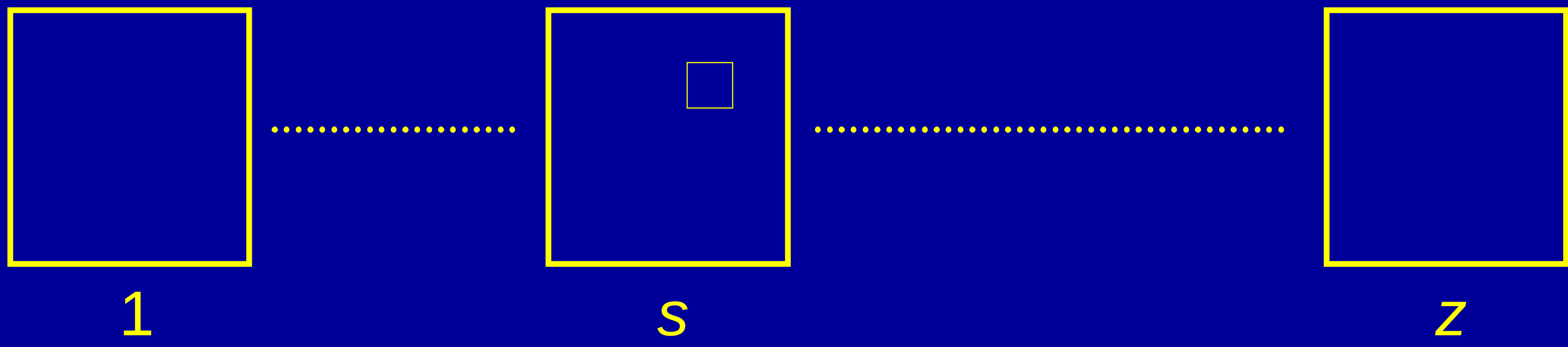


$r_s(a_s)$

$p_s(a_s)$

$a_s = (a^1_s\ , a^2_s\ ,\ L\ , a^n_s)$ joint action

Finite Stochastic Game



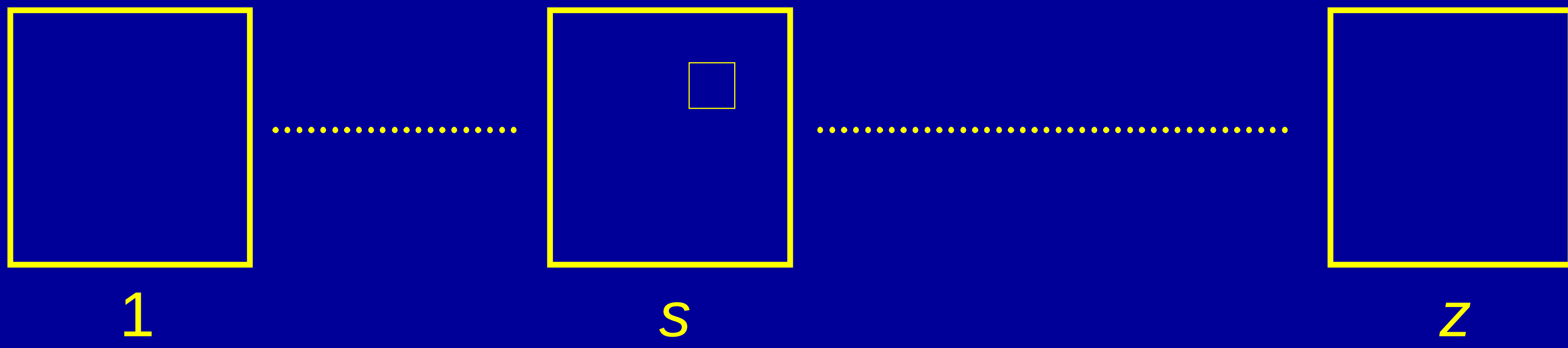
$r_s(a_s)$
$p_s(a_s)$

$a_s = (a^1_s\ ,\ a^2_s\ ,\ \mathrm{L}\ ,\ a^n_s)$ joint action

$r_s(a_s) = (r^1_s(a_s), r^2_s(a_s),\ L\ , r^n_s(a_s))$ rewards

$p_s(a_s) = (p_s(1|a_s), p_s(2|a_s),\ L\ , p_s(z|a_s))$ transitions

Finite Stochastic Game



- *Infinite horizon*
- *Complete Information*
- *Perfect Recall*
- *Independent and Simultaneous Choices*

3-Player Stochastic Game

State 1 — Player 3: N

1 \ 2	L	R
T	0, 0, 0 (1, 0, 0, 0)	0, 0, 0 (2/3, 0, 1/3, 0)
B	0, 0, 0 (2/3, 1/3, 0, 0)	0, 0, 0 (1/3, 1/3, 1/3, 0)

Player 3: F

	L	R
T	0, 0, 0 (2/3, 0, 0, 1/3)	0, 0, 0 (1/3, 0, 1/3, 1/3)
B	0, 0, 0 (1/3, 1/3, 0, 1/3)	0, 0, 0 (0, 1/3, 1/3, 1/3)

2	3	4
1, 3, 0 (0, 1, 0, 0)	0, 3, 1 (0, 0, 1, 0)	3, 0, 1 (0, 0, 0, 1)

Strategies

general strategy $\pi^i : N \times S \times H \to X^i$

$$(k, s, h) \to X^i_s$$

Strategies

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$(k, s, h) \to X^i_s$ mixed actions



Strategies

general strategy $\pi^i : N \times S \times H \to X^i$

$(k, s, h) \to X^i_s$ mixed actions

Markov strategy $f^i : N \times S \to X^i$

$(k, s) \to X^i_s$

Strategies

general strategy $\pi^i : N \times S \times H \rightarrow X^i$

$(k, s, h) \rightarrow X^i_s$ mixed actions

Markov strategy $f^i : N \times S \rightarrow X^i$

$(k, s) \rightarrow X^i_s$

stationary strategy $x^i : S \rightarrow X^i$

$(s) \rightarrow X^i_s$

Strategies

general strategy $\pi^i : N \times S \times H \to X^i$

$(k, s, h) \to X^i_s$ mixed actions

Markov strategy $f^i : N \times S \to X^i$

$(k, s) \to X^i_s$

stationary strategy $x^i : S \to X^i$

$(s) \to X^i_s$

opponents' strategy π^{-i} , f^{-i} *and* x^{-i}



Rewards

β-Discounted rewards (with $0 < \beta < 1$)

$$\gamma^i_{\beta s}(\pi) = E_{s\pi}((1-\beta) \sum_k \beta^{k-1} R^i_k)$$

Limiting average rewards

$$\gamma^i_s(\pi) = E_{s\pi}(\underline{\lim}_{K \to \infty} K^{-1} \sum^K_{k=1} R^i_k)$$



MinMax Values

β-Discounted minmax

$$v^i_{\beta s} = \inf_{\pi^{-i}} \sup_{\pi^i} \gamma^i_{\beta s}(\pi)$$

Limiting average minmax

$$v^i_s = \inf_{\pi^{-i}} \sup_{\pi^i} \gamma^i_s(\pi)$$

MinMax Values

β-Discounted minmax

$$v^i_{\beta s} = \inf_{\pi^{-i}} \sup_{\pi^i} \gamma^i_{\beta s}(\pi)$$

Limiting average minmax

$$v^i_s = \inf_{\pi^{-i}} \sup_{\pi^i} \gamma^i_s(\pi)$$

Highest rewards player *i* can defend

ε–Equilibrium

$\pi = (\pi^i)_{i \in N}$ is an ε-equilibrium if

$$\gamma^i_s(\sigma^i, \pi^{-i}) \leq \gamma^i_s(\pi) + \varepsilon$$

for all σ^i, for all i and for all s.

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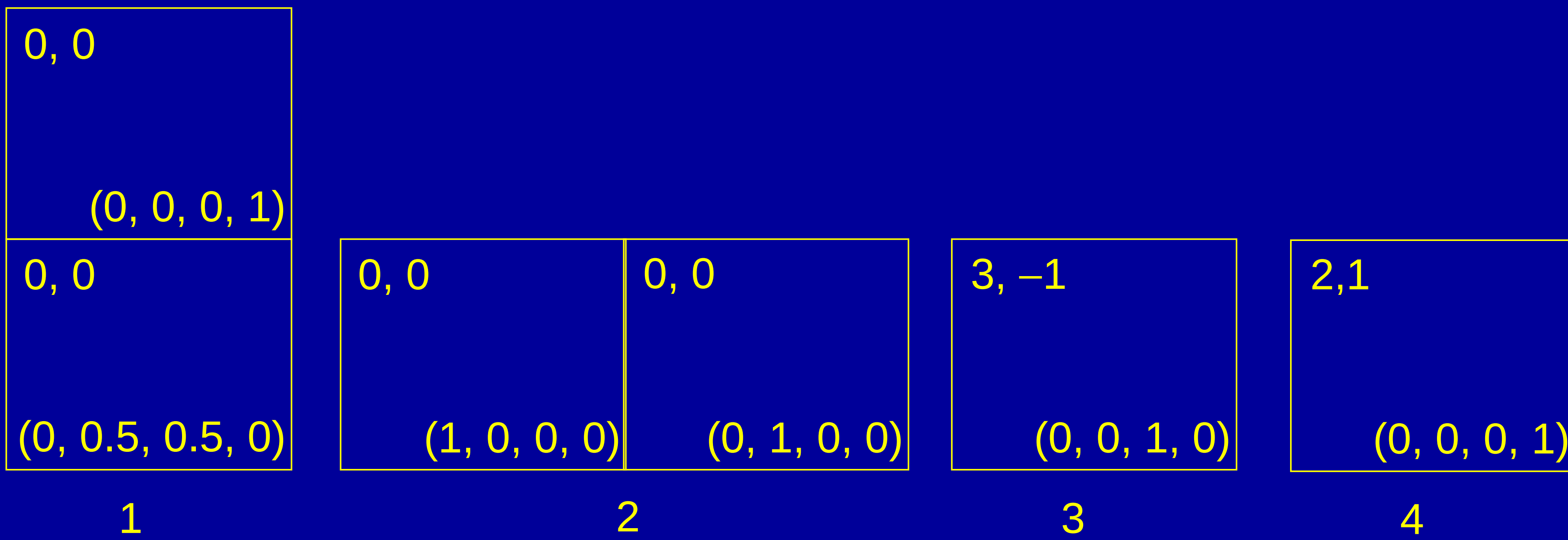
Question

Any ε-equilibrium?



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Highlights from (Finite) Stochastic Games History

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Shapley, 1953

0-sum, “discounted”

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0-sum, “discounted”

Everett, 1957

0-sum, recursive, undiscounted

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0-sum, “discounted”

Everett, 1957

0-sum, recursive, undiscounted

Gillette, 1957

0-sum, big match problem

Highlights from (Finite) Stochastic Games History

Fink, 1964 & Takahashi, 1964

n-player, discounted

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Fink, 1964 & Takahashi, 1964

n-player, discounted

Blackwell & Ferguson, 1968

0-sum, big match solution

Highlights from (Finite) Stochastic Games History

Fink, 1964 & Takahashi, 1964

n-player, discounted

Blackwell & Ferguson, 1968

0-sum, big match solution

Liggett & Lippmann, 1969

0-sum, perfect inf., undiscounted

Highlights from (Finite) Stochastic Games History

Kohlberg, 1974

0-sum, absorbing, undiscounted

Highlights from (Finite) Stochastic Games History

Kohlberg, 1974

0-sum, absorbing, undiscounted

Mertens & Neyman, 1981

0-sum, undiscounted

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Highlights from (Finite) Stochastic Games History

Kohlberg, 1974

0-sum, absorbing, undiscounted

Mertens & Neyman, 1981

0-sum, undiscounted

Sorin, 1986

2-player, Paris Match, undiscounted

Highlights from (Finite) Stochastic Games History

Vrieze & Thuijsman, 1989

2-player, absorbing, undiscounted

Highlights from (Finite) Stochastic Games History

Vrieze & Thuijsman, 1989

2-player, absorbing, undiscounted

Thuijsman & Raghavan, 1997

n-player, perfect inf., undiscounted

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Vrieze & Thuijsman, 1989

2-player, absorbing, undiscounted

Thuijsman & Raghavan, 1997

n-player, perfect inf., undiscounted

Flesch, Thuijsman, Vrieze, 1997

3-player, absorbing example, undiscounted

Highlights from (Finite) Stochastic Games History

Solan, 1999

3-player, absorbing, undiscounted

Highlights from (Finite) Stochastic Games History

Solan, 1999

3-player, absorbing, undiscounted

Vieille, 2000

2-player, undiscounted

Highlights from (Finite) Stochastic Games History

Solan, 1999

3-player, absorbing, undiscounted

Vieille, 2000

2-player, undiscounted

Solan & Vieille, 2001

n-player, quitting, undiscounted

Additive Transitions

$$p_s(a_s) = \sum^{n}_{i=1} \lambda^i_s \, p^i_s(a^i_s)$$

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$p^i_s(a^i_s)$ transition probabilities controlled by player i in state s

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$p^i_s(a^i_s)$ transition probabilities controlled by player i in state s

λ^i_s transition power of player i in state s

$0 \leq \lambda^i_s \leq 1$ and $\sum_i \lambda^i_s = 1$ for each s

Example for 2-Player Additive Transitions

$$p_s(a_s) = \sum^{n}_{i=1} \lambda^i_s \, p^i_s(a^i_s)$$

$\lambda^2_1 = 0.7$

$\lambda^1_1 = 0.3$

	$p^2_1(1)=(1, 0, 0)$	$p^2_1(2)=(0, 1, 0)$
$p^1_1(1) = (1, 0, 0)$		
$p^1_1(2) = (0, 0, 1)$		

1

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Example for 2-Player Additive Transitions

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$\lambda^2_1 = 0.7$

$\lambda^1_1 = 0.3$

	$p^2_1(1)=(1, 0, 0)$	$p^2_1(2)=(0, 1, 0)$
$p^1_1(1) = (1, 0, 0)$	(1, 0, 0)	(0.3, 0.7, 0)
$p^1_1(2) = (0, 0, 1)$	(0.7, 0, 0.3)	(0, 0.7, 0.3)

1

Results

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1. 0-equilibria for *n*-player AT games (threats!)

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2. 0-opt. stationary strat. for 0-sum AT games

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   neither to 3-player abs. AT games,
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Results

1. 0-equilibria for *n*-player AT games (threats!)
2. 0-opt. stationary strat. for 0-sum AT games
3. Stat. ε-equilibria for 2-player abs. AT games
4. Result 3 can not be strengthened,
   neither to 3-player abs. AT games,
   nor to 2-player non-abs. AT games,
   nor to give stationary 0-equilibria

The Essential Observation

Additive Transitions

induce

a Complete Ordering of the Actions

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Additive Transitions

induce

a Complete Ordering of the Actions

If a^i_s is “better” than b^i_s against some strategy, then a^i_s is “better” than b^i_s against any strategy.

"Better"

Consider strategies a^i_s and b^i_s for player i

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Consider strategies a^i_s and b^i_s for player i

If, for some strategy a^{-i}_s *we have*

$$\sum_{t \in S} p_s(t \mid a^i_s\ ,\ a^{-i}_s\ )\ v^i_t \geq \sum_{t \in S} p_s(t \mid b^i_s\ ,\ a^{-i}_s\ )\ v^i_t$$

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then for all strategies b^{-i}_s *we have*

$$\sum_{t \in S} p_s(t \mid a^i_s , b^{-i}_s) v^i_t \geq \sum_{t \in S} p_s(t \mid b^i_s , b^{-i}_s) v^i_t$$

Because ….

If $\sum_{t \in S} p_s(t \mid a^i_s, a^{-i}_s) v^i_t \geq \sum_{t \in S} p_s(t \mid b^i_s, a^{-i}_s) v^i_t$

Because ….

If $\sum_{t \in S} p_s(t \mid a^i_s , a^{-i}_s) v^i_t \geq \sum_{t \in S} p_s(t \mid b^i_s , a^{-i}_s) v^i_t$

Then

$$\lambda^i_s \sum_{t \in S} p_s(t \mid a^i_s)\, v^i_t + \sum_{j \neq i} \lambda^j_s \sum_{t \in S} p_s(t \mid a^{-j}_s)\, v^i_t \geq$$
$$\lambda^i_s \sum_{t \in S} p_s(t \mid b^i_s)\, v^i_t + \sum_{j \neq i} \lambda^j_s \sum_{t \in S} p_s(t \mid a^{-j}_s)\, v^i_t$$

Because ….

If $\sum_{t \in S} p_s(t \mid a^i_s\,, a^{-i}_s) v^i_t \geq \sum_{t \in S} p_s(t \mid b^i_s\,, a^{-i}_s) v^i_t$

Then

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Because ….

If $\sum_{t \in S} p_s(t \mid a^i_s, a^{-i}_s) v^i_t \geq \sum_{t \in S} p_s(t \mid b^i_s, a^{-i}_s) v^i_t$

Then

$$\lambda^i_s \sum_{t \in S} p_s(t \mid a^i_s)\, v^i_t + \sum_{j \neq i} \lambda^j_s \sum_{t \in S} p_s(t \mid a^{-j}_s)\, v^i_t \geq$$
$$\lambda^i_s \sum_{t \in S} p_s(t \mid b^i_s)\, v^i_t + \sum_{j \neq i} \lambda^j_s \sum_{t \in S} p_s(t \mid a^{-j}_s)\, v^i_t$$

which implies that ….

$$\lambda^i_s \sum_{t \in S} p_s(t \mid a^i_s)\, v^i_t + \sum_{j \neq i} \lambda^j_s \sum_{t \in S} p_s(t \mid b^{-j}_s)\, v^i_t \geq$$
$$\lambda^i_s \sum_{t \in S} p_s(t \mid b^i_s)\, v^i_t + \sum_{j \neq i} \lambda^j_s \sum_{t \in S} p_s(t \mid b^{-j}_s)\, v^i_t$$

which implies that ….

$$\boxed{\lambda^i_s \sum_{t \in S} p_s(t \mid a^i_s)\, v^i_t} + \sum_{j \neq i} \lambda^j_s \sum_{t \in S} p_s(t \mid b^{-j}_s)\, v^i_t \geq$$
$$\boxed{\lambda^i_s \sum_{t \in S} p_s(t \mid b^i_s)\, v^i_t} + \sum_{j \neq i} \lambda^j_s \sum_{t \in S} p_s(t \mid b^{-j}_s)\, v^i_t$$

And therefore

$$\sum_{t \in S} p_s(t \mid a^i_s\,,\, b^{-i}_s) v^i_t \geq \sum_{t \in S} p_s(t \mid b^i_s\,,\, b^{-i}_s) v^i_t$$

“Best”

The “best” actions for player *i* in state *s* are those that maximize the expression

$$\sum_{t \in S} p_s(t \mid a^i_s)\, v^i_t$$

The Restricted Game

Let *G* be the original AT game and let *G** be the restricted AT game, where each player is restricted to his “best” actions.

The Restricted Game

Now $v^{*i} \geq v^i$ for each player i.

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In G^* : $\sum_{t \in S} p_s(t \mid a^*_s)\, v^{*i}_t = v^{*i}_s \qquad \forall\, i, s, a^*_s$

The Restricted Game

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In G^* : $\sum_{t \in S} p_s(t \mid a^*_s)\, v^{*i}_t = v^{*i}_s \qquad \forall\, i, s, a^*_s$

In G : $\sum_{t \in S} p_s(t \mid b^i_s, a^{*-i}_s)\, v^i_t < v^i_s \quad \forall\, i, s,$

$\forall\, a^{*-i}_s, b^i_s$

The Restricted Game

If x^{*i} *yields at least* v^{*i} *in G*,*

then x^{*i} *yields at least* v^{*i} *in G as well.*

Ex. 1: 2-Player Absorbing AT Game

0, 0 (1, 0, 0)	0, 0 (0.5, 0.5, 0)
0, 0 (0.5, 0, 0.5)	0, 0 (0, 0.5, 0.5)

1

–3, 1 (0, 1, 0)

2

–1, 3 (0, 0, 1)

3

Ex. 1: 2-Player Absorbing AT Game

0.5

	(1, 0, 0)	(0, 1, 0)
(1, 0, 0)	0, 0 — (1, 0, 0)	0, 0 — (0.5, 0.5, 0)
0.5 (0, 0, 1)	0, 0 — (0.5, 0, 0.5)	0, 0 — (0, 0.5, 0.5)

1

–3, 1 — (0, 1, 0)

2

–1, 3 — (0, 0, 1)

3

Ex. 1: 2-Player Absorbing AT Game

NO stationary 0-equilibrium

State 1:

0, 0 (1, 0, 0)	0, 0 (0.5, 0.5, 0)
0, 0 (0.5, 0, 0.5)	0, 0 (0, 0.5, 0.5)

1

State 2:

–3, 1 (0, 1, 0)

2

State 3:

–1, 3 (0, 0, 1)

3

Ex. 1: 2-Player Absorbing AT Game

NO stationary 0-equilibrium

State 1:

	0, 0 (1, 0, 0)	0, 0 (0.5, 0.5, 0)
	0, 0 (0.5, 0, 0.5)	0, 0 (0, 0.5, 0.5)

State 2:

–3, 1 (0, 1, 0)

State 3:

–1, 3 (0, 0, 1)

T, L

0, 0

Ex. 1: 2-Player Absorbing AT Game

NO stationary 0-equilibrium

State 1:

0, 0 (1, 0, 0)	0, 0 (0.5, 0.5, 0)
0, 0 (0.5, 0, 0.5)	0, 0 (0, 0.5, 0.5)

State 2:

–3, 1 (0, 1, 0)

State 3:

–1, 3 (0, 0, 1)

T, L	→	T, R
0, 0		-3, 1

Ex. 1: 2-Player Absorbing AT Game

NO stationary 0-equilibrium

State 1:

	0, 0 (1, 0, 0)	0, 0 (0.5, 0.5, 0)
	0, 0 (0.5, 0, 0.5)	0, 0 (0, 0.5, 0.5)

State 2:

–3, 1 (0, 1, 0)

State 3:

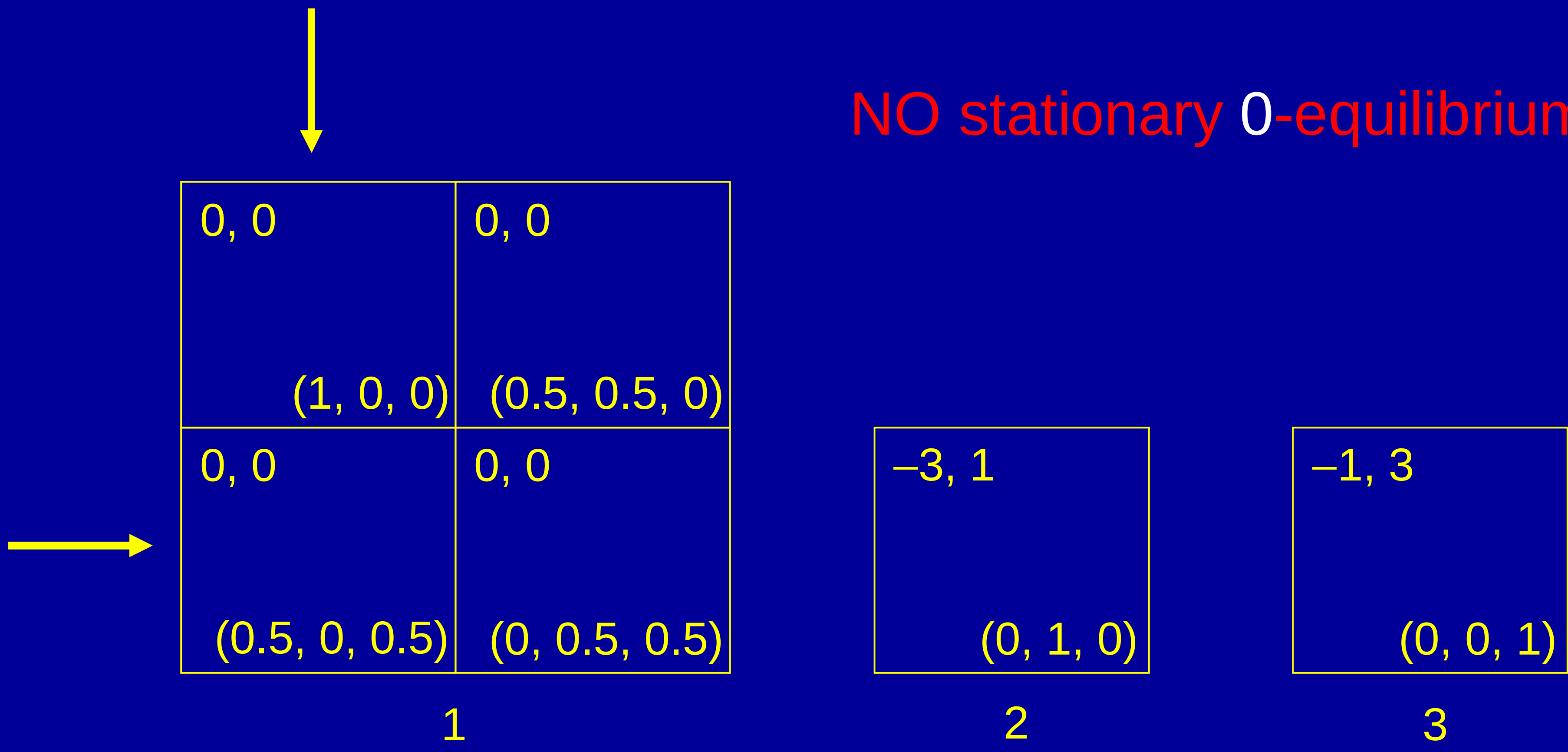
–1, 3 (0, 0, 1)

1 2 3

T, L	→	T, R	→	B, R
0, 0		-3, 1		-2, 2

Ex. 1: 2-Player Absorbing AT Game

NO stationary 0-equilibrium



T, L → T, R → B, R → B, L

0, 0 → -3, 1 → -2, 2 → -1, 3

Ex. 1: 2-Player Absorbing AT Game

NO stationary 0-equilibrium

State 1:

→	0, 0 (1, 0, 0)	0, 0 (0.5, 0.5, 0)
	0, 0 (0.5, 0, 0.5)	0, 0 (0, 0.5, 0.5)

State 2:

–3, 1 (0, 1, 0)

State 3:

–1, 3 (0, 0, 1)

T, L	→	T, R	→	B, R	→	B, L	→	T, L
0, 0		-3, 1		-2, 2		-1, 3		0, 0

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Ex. 1: 2-Player Absorbing AT Game

stationary ε-equilibrium

with $\varepsilon > 0$

0, 0 (1, 0, 0)	0, 0 (0.5, 0.5, 0)
0, 0 (0.5, 0, 0.5)	0, 0 (0, 0.5, 0.5)

1

–3, 1 (0, 1, 0)

2

–1, 3 (0, 0, 1)

3

Ex. 1: 2-Player Absorbing AT Game

	$1-\varepsilon/2$	$\varepsilon/2$
0	0, 0 (1, 0, 0)	0, 0 (0.5, 0.5, 0)
1	0, 0 (0.5, 0, 0.5)	0, 0 (0, 0.5, 0.5)

1

stationary ε-equilibrium

with $\varepsilon > 0$

–3, 1 (0, 1, 0)

2

–1, 3 (0, 0, 1)

3

Ex. 1: 2-Player Absorbing AT Game

	$1-\varepsilon/2$	$\varepsilon/2$
0	0, 0 (1, 0, 0)	0, 0 (0.5, 0.5, 0)
1	0, 0 (0.5, 0, 0.5)	0, 0 (0, 0.5, 0.5)

1

–3, 1 (0, 1, 0)

2

–1, 3 (0, 0, 1)

3

stationary ε-equilibrium

with $\varepsilon > 0$

equilibrium rewards ≈ $((-1-\varepsilon, 3-\varepsilon), (-3,1), (-1,3))$

Ex. 1: 2-Player Absorbing AT Game

non-stationary 0-equilibrium

State 1:

0, 0 (1, 0, 0)	0, 0 (0.5, 0.5, 0)
0, 0 (0.5, 0, 0.5)	0, 0 (0, 0.5, 0.5)

1

State 2:

–3, 1 (0, 1, 0)

2

State 3:

–1, 3 (0, 0, 1)

3

Ex. 1: 2-Player Absorbing AT Game

non-stationary 0-equilibrium

Player 1: B, T, T, T, ….

Player 2: R, R, R, R, ….

	0, 0 (1, 0, 0)	0, 0 (0.5, 0.5, 0)
	0, 0 (0.5, 0, 0.5)	0, 0 (0, 0.5, 0.5)

1

–3, 1 (0, 1, 0)

2

–1, 3 (0, 0, 1)

3

Ex. 1: 2-Player Absorbing AT Game

	0, 0 (1, 0, 0)	0, 0 (0.5, 0.5, 0)
→	0, 0 (0.5, 0, 0.5)	0, 0 (0, 0.5, 0.5)

1

–3, 1 (0, 1, 0)

2

–1, 3 (0, 0, 1)

3

non-stationary 0-equilibrium

Player 1: B, T, T, T, ….

Player 2: R, R, R, R, ….

Ex. 1: 2-Player Absorbing AT Game

non-stationary 0-equilibrium

Player 1: B, T, T, T, ….

Player 2: R, R, R, R, ….

State 1:

0, 0 (1, 0, 0)	0, 0 (0.5, 0.5, 0)
0, 0 (0.5, 0, 0.5)	0, 0 (0, 0.5, 0.5)

State 2:

–3, 1 (0, 1, 0)

State 3:

–1, 3 (0, 0, 1)

Ex. 1: 2-Player Absorbing AT Game

non-stationary 0-equilibrium

Player 1: B, T, T, T, ….

Player 2: R, R, R, R, ….

1

0, 0 (1, 0, 0)	0, 0 (0.5, 0.5, 0)
0, 0 (0.5, 0, 0.5)	0, 0 (0, 0.5, 0.5)

2

–3, 1 (0, 1, 0)

3

–1, 3 (0, 0, 1)

equilibrium rewards ((–2, 2), (–3, 1), (–1, 3))

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Ex. 2: 2-Player Non-Absorbing AT Game

NO stationary ε-equilibrium with $\varepsilon > 0$

State 1

0, 0	(0, 0, 0, 1)
0, 0	(0, 0.5, 0.5, 0)

State 2

0, 0	(1, 0, 0, 0)	0, 0	(0, 1, 0, 0)

State 3

3, –1	(0, 0, 1, 0)

State 4

2,1	(0, 0, 0, 1)

Ex. 2: 2-Player Non-Absorbing AT Game

NO stationary ε-equilibrium

with $\varepsilon > 0$

State 1

p	0, 0 — (0, 0, 0, 1)
$1-p$	0, 0 — (0, 0.5, 0.5, 0)

State 2

q	$1-q$
0, 0 — (1, 0, 0, 0)	0, 0 — (0, 1, 0, 0)

State 3

3, –1 — (0, 0, 1, 0)

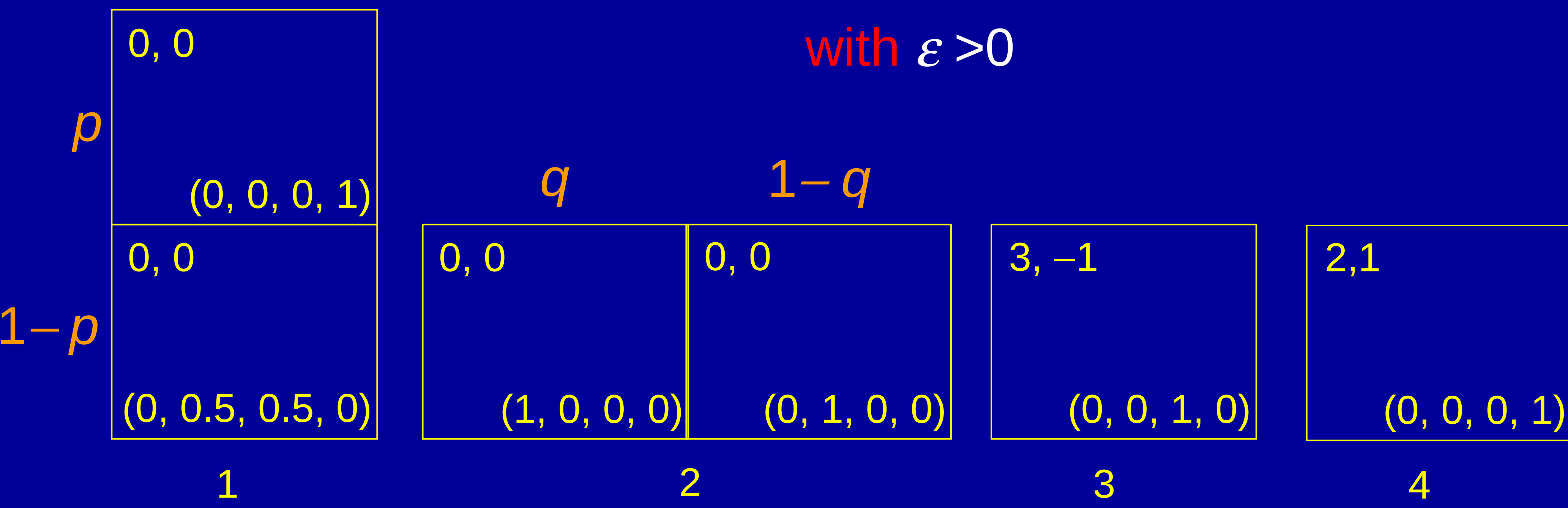
State 4

2,1 — (0, 0, 0, 1)

Ex. 2: 2-Player Non-Absorbing AT Game

NO stationary ε-equilibrium

with $\varepsilon > 0$



$q > 0$

Ex. 2: 2-Player Non-Absorbing AT Game

NO stationary ε-equilibrium

with $\varepsilon > 0$

State 1:

p	0, 0 (0, 0, 0, 1)
$1-p$	0, 0 (0, 0.5, 0.5, 0)

State 2:

q	$1-q$
0, 0 (1, 0, 0, 0)	0, 0 (0, 1, 0, 0)

State 3:

3, –1 (0, 0, 1, 0)

State 4:

2,1 (0, 0, 0, 1)

1 2 3 4

$q > 0 \longrightarrow p < \varepsilon$



Ex. 2: 2-Player Non-Absorbing AT Game

NO stationary ε-equilibrium

with $\varepsilon > 0$

State 1:

p	0, 0 — (0, 0, 0, 1)
$1-p$	0, 0 — (0, 0.5, 0.5, 0)

State 2:

q	$1-q$
0, 0 — (1, 0, 0, 0)	0, 0 — (0, 1, 0, 0)

State 3:

3, –1 — (0, 0, 1, 0)

State 4:

2,1 — (0, 0, 0, 1)

$q > 0 \longrightarrow p < \varepsilon \longrightarrow q = 0$

Ex. 2: 2-Player Non-Absorbing AT Game

NO stationary ε-equilibrium

with $\varepsilon > 0$

	State 1		State 2: q	State 2: $1-q$	State 3	State 4
p	0, 0 (0, 0, 0, 1)					
$1-p$	0, 0 (0, 0.5, 0.5, 0)		0, 0 (1, 0, 0, 0)	0, 0 (0, 1, 0, 0)	3, –1 (0, 0, 1, 0)	2,1 (0, 0, 0, 1)

1 2 3 4

$$q > 0 \longrightarrow p < \varepsilon \longrightarrow q = 0 \longrightarrow p > 1 - \varepsilon$$

Ex. 2: 2-Player Non-Absorbing AT Game

NO stationary ε-equilibrium

with $\varepsilon > 0$

State 1:

p	0, 0 — (0, 0, 0, 1)
$1-p$	0, 0 — (0, 0.5, 0.5, 0)

State 2:

q	$1-q$
0, 0 — (1, 0, 0, 0)	0, 0 — (0, 1, 0, 0)

State 3:

3, –1 — (0, 0, 1, 0)

State 4:

2,1 — (0, 0, 0, 1)

$$q > 0 \longrightarrow p < \varepsilon \longrightarrow q = 0 \longrightarrow p > 1-\varepsilon \longrightarrow q > 0$$

Ex. 2: 2-Player Non-Absorbing AT Game

non-stationary 0-equilibrium

State 1:

0, 0 (0, 0, 0, 1)
0, 0 (0, 0.5, 0.5, 0)

State 2:

0, 0 (1, 0, 0, 0)	0, 0 (0, 1, 0, 0)

State 3:

3, –1 (0, 0, 1, 0)

State 4:

2,1 (0, 0, 0, 1)

Ex. 2: 2-Player Non-Absorbing AT Game

non-stationary 0-equilibrium

Player 1: T, B, B, B, ….

Player 2: R, R, R, R, ….

0, 0 (0, 0, 0, 1)
0, 0 (0, 0.5, 0.5, 0)

1

0, 0 (1, 0, 0, 0)	0, 0 (0, 1, 0, 0)

2

3, –1 (0, 0, 1, 0)

3

2,1 (0, 0, 0, 1)

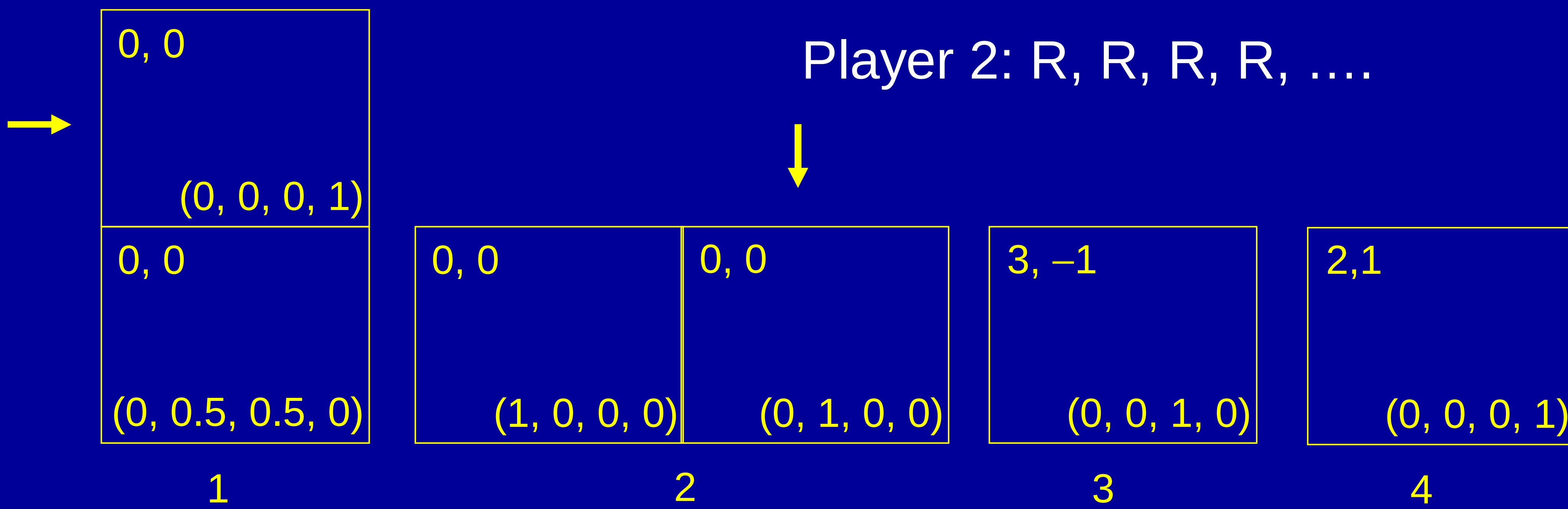
4

Ex. 2: 2-Player Non-Absorbing AT Game

non-stationary 0-equilibrium

Player 1: T, B, B, B, ….

Player 2: R, R, R, R, ….

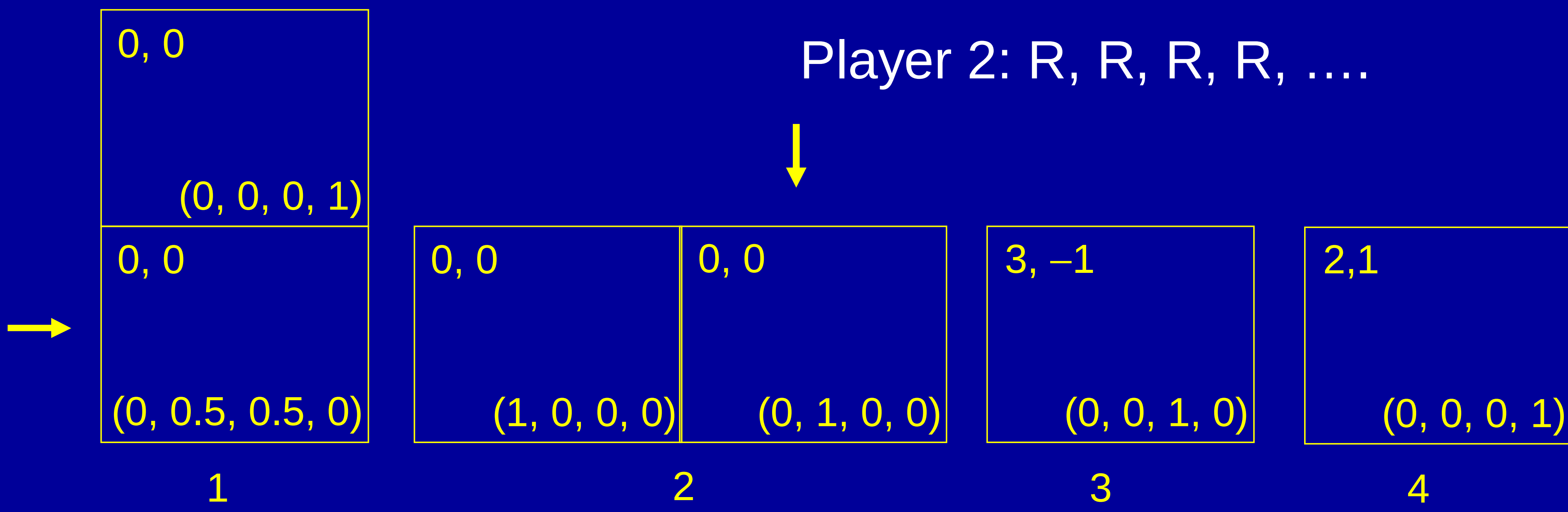


Ex. 2: 2-Player Non-Absorbing AT Game

non-stationary 0-equilibrium

Player 1: T, B, B, B, ….

Player 2: R, R, R, R, ….



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Ex. 2: 2-Player Non-Absorbing AT Game

non-stationary 0-equilibrium

Player 1: T, B, B, B, ….

Player 2: R, R, R, R, ….

State 1:

0, 0 (0, 0, 0, 1)	
0, 0 (0, 0.5, 0.5, 0)	

State 2:

0, 0 (1, 0, 0, 0)	0, 0 (0, 1, 0, 0)

State 3:

3, –1 (0, 0, 1, 0)

State 4:

2,1 (0, 0, 0, 1)

1 2 3 4

equilibrium rewards ((2, 1), (0, 0), (3, –1), 2, 1))

Ex. 2: 2-Player Non-Absorbing AT Game

alternative 0-equilibrium

Player 1: T, T, T, T, ….

Player 2: L, R, R, R, ….

State 1	State 2 (L)	State 2 (R)	State 3	State 4
0, 0 (0, 0, 0, 1)				
0, 0 (0, 0.5, 0.5, 0)	0, 0 (1, 0, 0, 0)	0, 0 (0, 1, 0, 0)	3, –1 (0, 0, 1, 0)	2,1 (0, 0, 0, 1)

1 2 3 4

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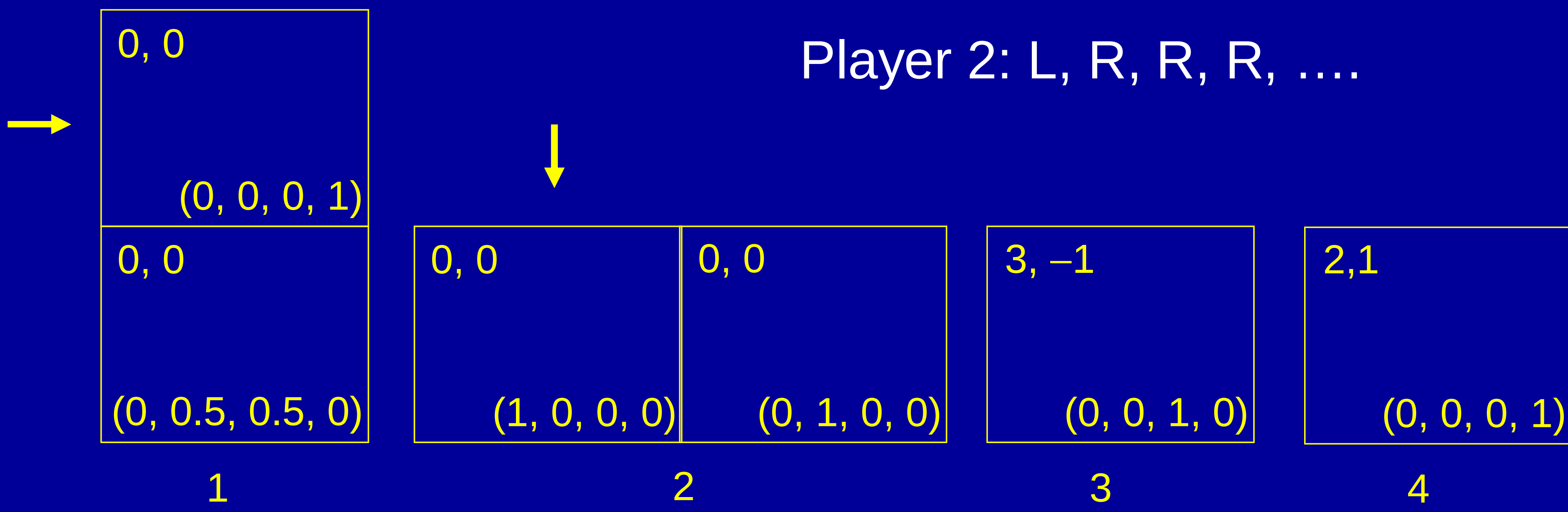
Ex. 2: 2-Player Non-Absorbing AT Game

alternative 0-equilibrium

Player 1: T, T, T, T, ….

Player 2: L, R, R, R, ….

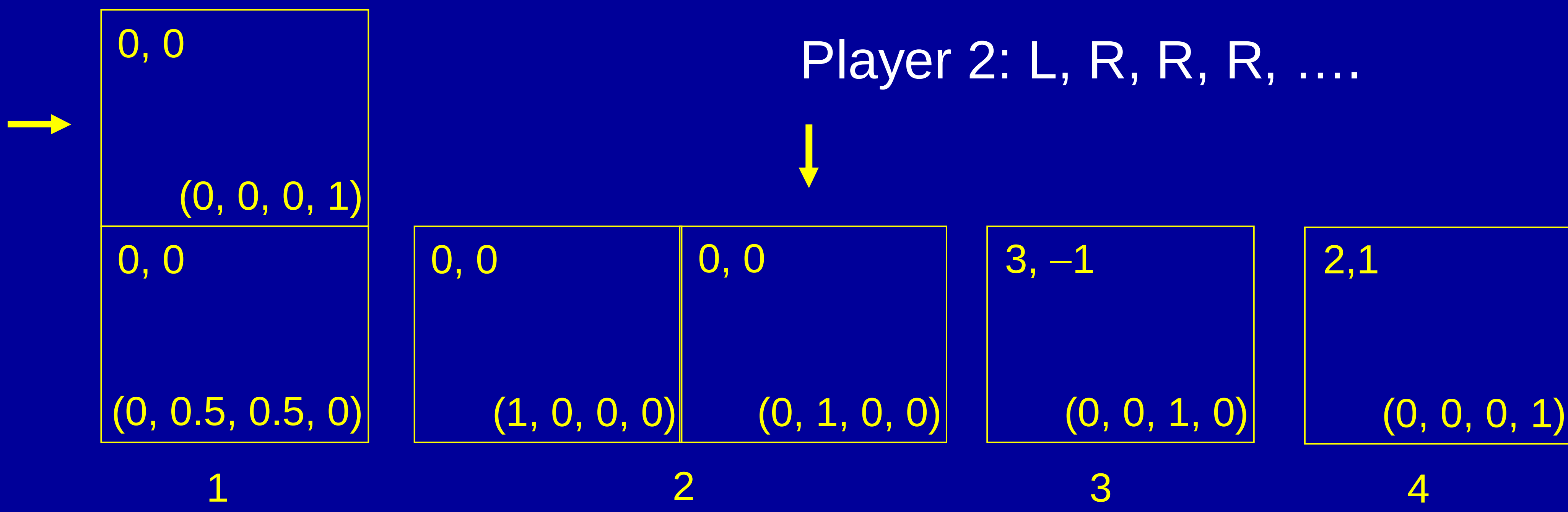


Ex. 2: 2-Player Non-Absorbing AT Game

alternative 0-equilibrium

Player 1: T, T, T, T, ….

Player 2: L, R, R, R, ….



Ex. 2: 2-Player Non-Absorbing AT Game

alternative 0-equilibrium

Player 1: T, T, T, T, ….

Player 2: L, R, R, R, ….

State 1:

0, 0 (0, 0, 0, 1)
0, 0 (0, 0.5, 0.5, 0)

State 2:

0, 0 (1, 0, 0, 0)	0, 0 (0, 1, 0, 0)

State 3:

3, –1 (0, 0, 1, 0)

State 4:

2,1 (0, 0, 0, 1)

equilibrium rewards ((2, 1), (2, 1), (3, –1), 2, 1))

Ex. 3: 3-Player Absorbing AT Game

3

N

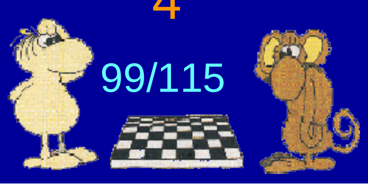
1 \ 2	L	R
T	0, 0, 0 (1, 0, 0, 0)	0, 0, 0 (2/3, 0, 1/3, 0)
B	0, 0, 0 (2/3, 1/3, 0, 0)	0, 0, 0 (1/3, 1/3, 1/3, 0)

F

	0, 0, 0 (2/3, 0, 0, 1/3)	0, 0, 0 (1/3, 0, 1/3, 1/3)
	0, 0, 0 (1/3, 1/3, 0, 1/3)	0, 0, 0 (0, 1/3, 1/3, 1/3)

1

2	3	4
1, 3, 0 (0, 1, 0, 0)	0, 1, 3 (0, 0, 1, 0)	3, 0, 1 (0, 0, 0, 1)



Ex. 3: 3-Player Absorbing AT Game

Player 3: N

1 \ 2	L	R
T	0, 0, 0 (1, 0, 0, 0)	0, 0, 0 (2/3, 0, 1/3, 0)
B	0, 0, 0 (2/3, 1/3, 0, 0)	0, 0, 0 (1/3, 1/3, 1/3, 0)

Player 3: F

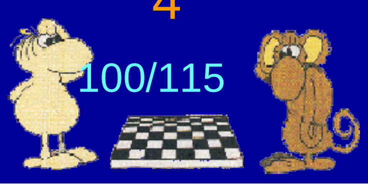
T	0, 0, 0 (2/3, 0, 0, 1/3)	0, 0, 0 (1/3, 0, 1/3, 1/3)
B	0, 0, 0 (1/3, 1/3, 0, 1/3)	0, 0, 0 (0, 1/3, 1/3, 1/3)

1

2	3	4
1, 3, 0 (0, 1, 0, 0)	0, 1, 3 (0, 0, 1, 0)	3, 0, 1 (0, 0, 0, 1)

How to share 4 among three people

if only 3 solutions are allowed?



Ex. 3: 3-Player Absorbing AT Game

N

	L	R
T	0, 0, 0 (1, 0, 0, 0)	0, 1, 3 (2/3, 0, 1/3, 0)
B	1, 3, 0 (2/3, 1/3, 0, 0)	1/2, 2, 3/2 (1/3, 1/3, 1/3, 0)

F

	3, 0, 1 (2/3, 0, 0, 1/3)	3/2, 1/2, 2 (1/3, 0, 1/3, 1/3)
	2, 3/2, 1/2 (1/3, 1/3, 0, 1/3)	4/3, 4/3, 4/3 (0, 1/3, 1/3, 1/3)

1

State	Payoff	Transition
2	1, 3, 0	(0, 1, 0, 0)
3	0, 1, 3	(0, 0, 1, 0)
4	3, 0, 1	(0, 0, 0, 1)

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Ex. 3: 3-Player Absorbing AT Game

N

	L	R
T	0, 0, 0	0, 1, 3 1/3 *
B	1, 3, 0 1/3 *	1/2, 2, 3/2 2/3 *

F

	L	R
T	3, 0, 1 1/3 *	3/2, 1/2, 2 2/3 *
B	2, 3/2, 1/2 2/3 *	4/3, 4/3, 4/3 1 *

Ex. 3: 3-Player Absorbing AT Game

N

	L	R
T	0, 0, 0	0, 1, 3 1/3 *
B	1, 3, 0 1/3 *	1/2, 2, 3/2 2/3 *

F

T	3, 0, 1 1/3 *	3/2, 1/2, 2 2/3 *
B	2, 3/2, 1/2 2/3 *	4/3, 4/3, 4/3 1 *

NO stationary ε-equilibrium

Ex. 3: 3-Player Absorbing AT Game

N

	L	R
T	0, 0, 0	0, 1, 3 1/3 *
B	1, 3, 0 1/3 *	1/2, 2, 3/2 2/3 *

F

T	3, 0, 1 1/3 *	3/2, 1/2, 2 2/3 *
B	2, 3/2, 1/2 2/3 *	4/3, 4/3, 4/3 1 *

NO stationary ε-equilibrium

Ex. 3: 3-Player Absorbing AT Game

N

	L	R
T	0, 0, 0	0, 1, 3 1/3 *
B	1, 3, 0 1/3 *	1/2, 2, 3/2 2/3 *

F

3, 0, 1 1/3 *	3/2, 1/2, 2 2/3 *
2, 3/2, 1/2 2/3 *	4/3, 4/3, 4/3 1 *

non-stationary 0-equilibrium

Ex. 3: 3-Player Absorbing AT Game

N

	L	R
T	0, 0, 0	0, 1, 3 — 1/3 *
B	1, 3, 0 — 1/3 *	1/2, 2, 3/2 — 2/3 *

F

	L	R
T	3, 0, 1 — 1/3 *	3/2, 1/2, 2 — 2/3 *
B	2, 3/2, 1/2 — 2/3 *	4/3, 4/3, 4/3 — 1 *

non-stationary 0-equilibrium

Player 1 on B: 1, ¾, 0, 0, 0, 0, 1, ¾, 0, 0, 0, 0, ….

Ex. 3: 3-Player Absorbing AT Game

N

	L	R
T	0, 0, 0	0, 1, 3 (1/3 *)
B	1, 3, 0 (1/3 *)	1/2, 2, 3/2 (2/3 *)

F

	L	R
T	3, 0, 1 (1/3 *)	3/2, 1/2, 2 (2/3 *)
B	2, 3/2, 1/2 (2/3 *)	4/3, 4/3, 4/3 (1 *)

non-stationary 0-equilibrium

Player 1 on B: 1, ¾, 0, 0, 0, 0, 1, ¾, 0, 0, 0, 0, ….

Player 2 on R: 0, 0, 1, ¾, 0, 0, 0, 0, 1, ¾, 0, 0, ….

Ex. 3: 3-Player Absorbing AT Game

N

	L	R
T	0, 0, 0	0, 1, 3 1/3 *
B	1, 3, 0 1/3 *	1/2, 2, 3/2 2/3 *

F

	3, 0, 1 1/3 *	3/2, 1/2, 2 2/3 *
	2, 3/2, 1/2 2/3 *	4/3, 4/3, 4/3 1 *

non-stationary 0-equilibrium

Player 1 on B: 1, ¾, 0, 0, 0, 0, 1, ¾, 0, 0, 0, 0, ….

Player 2 on R: 0, 0, 1, ¾, 0, 0, 0, 0, 1, ¾, 0, 0, ….

Player 3 on F: 0, 0, 0, 0, 1, ¾, 0, 0, 0, 0, 1, ¾, ….

Ex. 3: 3-Player Absorbing AT Game

N

	L	R
T	0, 0, 0	0, 1, 3 — 1/3 *
B	1, 3, 0 — 1/3 *	1/2, 2, 3/2 — 2/3 *

F

	L	R
T	3, 0, 1 — 1/3 *	3/2, 1/2, 2 — 2/3 *
B	2, 3/2, 1/2 — 2/3 *	4/3, 4/3, 4/3 — 1 *

non-stationary 0-equilibrium

Player 1 on B: 1, ¾, 0, 0, 0, 0, 1, ¾, 0, 0, 0, 0, ….

Player 2 on R: 0, 0, 1, ¾, 0, 0, 0, 0, 1, ¾, 0, 0, ….

Player 3 on F: 0, 0, 0, 0, 1, ¾, 0, 0, 0, 0, 1, ¾, ….

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Ex. 3: 3-Player Absorbing AT Game

N

	L	R
T	0, 0, 0	0, 1, 3 — 1/3 *
B	1, 3, 0 — 1/3 *	1/2, 2, 3/2 — 2/3 *

F

	L	R
T	3, 0, 1 — 1/3 *	3/2, 1/2, 2 — 2/3 *
B	2, 3/2, 1/2 — 2/3 *	4/3, 4/3, 4/3 — 1 *

non-stationary 0-equilibrium

Player 1 on B: 1, ¾, 0, 0, 0, 0, 1, ¾, 0, 0, 0, 0, ….

Player 2 on R: 0, 0, 1, ¾, 0, 0, 0, 0, 1, ¾, 0, 0, ….

Player 3 on F: 0, 0, 0, 0, 1, ¾, 0, 0, 0, 0, 1, ¾, ….

Ex. 3: 3-Player Absorbing AT Game

N

	L	R
T	0, 0, 0	0, 1, 3 — 1/3 *
B	1, 3, 0 — 1/3 *	1/2, 2, 3/2 — 2/3 *

F

	L	R
T	3, 0, 1 — 1/3 *	3/2, 1/2, 2 — 2/3 *
B	2, 3/2, 1/2 — 2/3 *	4/3, 4/3, 4/3 — 1 *

equilibrium rewards (1, 2, 1)

Player 1 on B: 1, ¾, 0, 0, 0, 0, 1, ¾, 0, 0, 0, 0, ….

Player 2 on R: 0, 0, 1, ¾, 0, 0, 0, 0, 1, ¾, 0, 0, ….

Player 3 on F: 0, 0, 0, 0, 1, ¾, 0, 0, 0, 0, 1, ¾, ….

Ex. 3: 3-Player Absorbing AT Game

N

	L	R
T	0, 0, 0	0, 1, 3 1/3 *
B	1, 3, 0 1/3 *	1/2, 2, 3/2 2/3 *

F

3, 0, 1 1/3 *	3/2, 1/2, 2 2/3 *
2, 3/2, 1/2 2/3 *	4/3, 4/3, 4/3 1 *

What is the simple 0-equilibrium using threats?

Ex. 3: 3-Player Absorbing AT Game

N

	L	R
T	0, 0, 0	0, 1, 3 1/3 *
B	1, 3, 0 1/3 *	1/2, 2, 3/2 2/3 *

F

	L	R
T	3, 0, 1 1/3 *	3/2, 1/2, 2 2/3 *
B	2, 3/2, 1/2 2/3 *	4/3, 4/3, 4/3 1 *

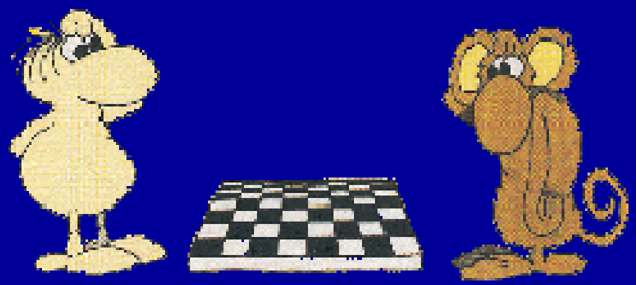
What is the simple 0-equilibrium using threats?

It is (B, R, F) with threats (R, N) to 1, (T, F) to 2, (B, L) to 3 !

Results

1. 0-equilibria for *n*-player AT games (threats!)
2. 0-opt. stationary strat. for 0-sum AT games
3. Stat. ε-equilibria for 2-player abs. AT games
4. Result 3 can not be strengthened,
   neither to 3-player abs. AT games,
   nor to 2-player non-abs. AT games,
   nor to give stat. 0-equilibria

?



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GAME  VER