One-Way Flow Nash Networks

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Joint work with
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MCNS Maastricht Center for Networks and Systems

Universiteit Maastricht

Outline

- The Model of One-Way Flow Networks
- An Existence Result
- A Counterexample

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The Model of One-Way Flow Networks

Network Formation Game (N, v, c)

- $N = \{1, 2, 3, \ldots, n\}$
- v_{ij} is the profit for agent i for being connected to agent j
- c_{ij} is the cost for agent i for making a link to agent j

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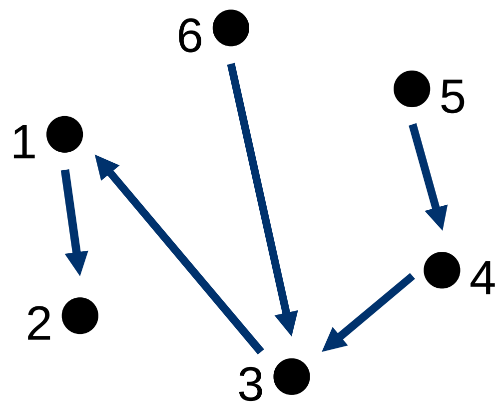
Example of a network g



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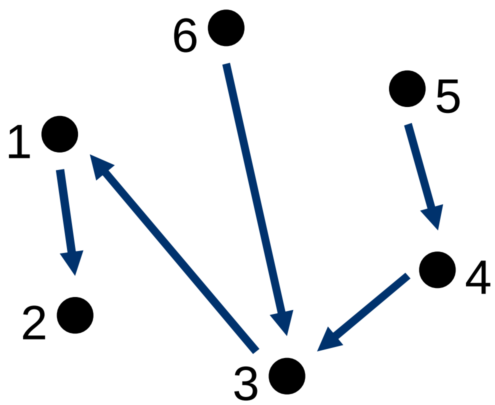
Agent 1 is connected to agents 3,4,5 and 6 and obtains profits v_{13} , v_{14} , v_{15} , v_{16}



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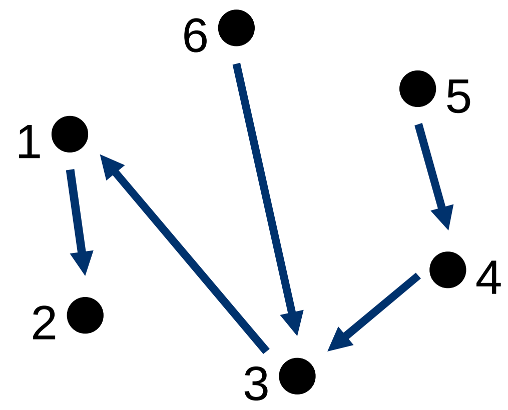

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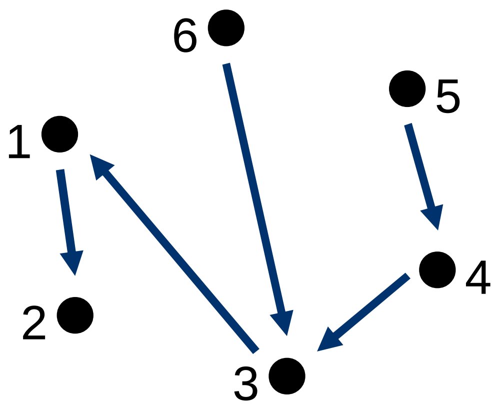

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Agent 1 has to pay c_{13} for the link (3,1)

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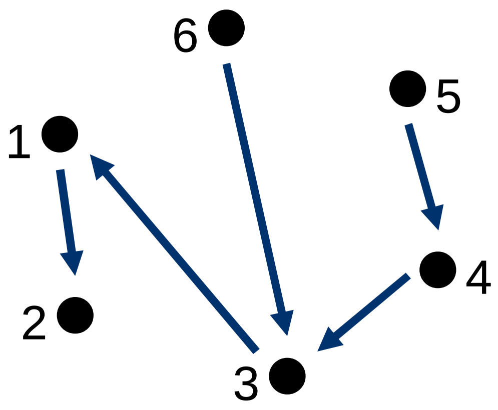


The payoff $\pi_1(g)$ for agent 1 is

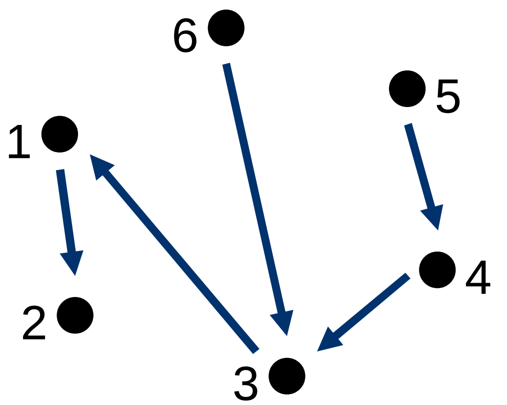
$$\pi_1(g) = v_{13} + v_{14} + v_{15} + v_{16} - c_{13}$$

The Model of One-Way Flow Networks

More generally

$$\pi_i(g) = \Sigma_{j \in N_i(g)} v_{ij} - \Sigma_{j \in Nd_i(g)} c_{ij}$$

where $N_i(g)$ is the set of agents that i is connected to, and
where $Nd_i(g)$ is the set of agents that i is directly connected to.



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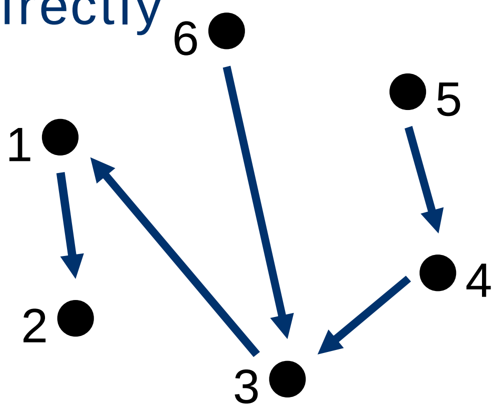
An action for agent i is any subset S of $N \setminus \{i\}$
indicating the set of agents that i connects to directly



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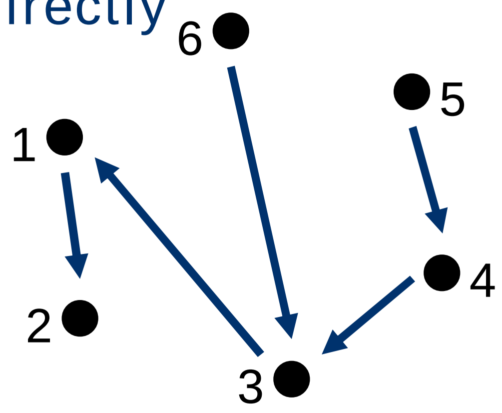


A network g is a Nash network
if each agent i is playing a best response
in terms of his individual payoff $\pi_i(g)$

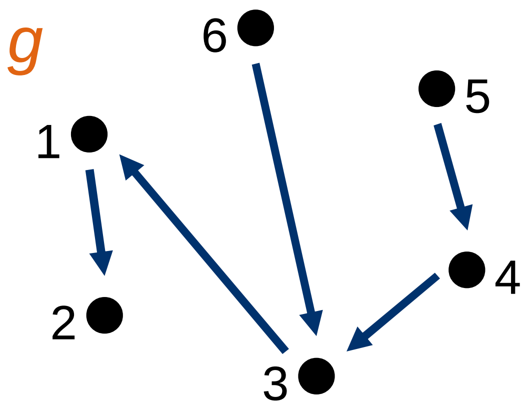
A Closer Look at Nash Networks

A network g is a Nash network if for each agent i

$$\boldsymbol{\pi}_i(g) \geq \boldsymbol{\pi}_i(g_{-i} + \{(j,i) : j \in S\})$$

for all subsets S of $N \setminus \{i\}$

Here g_{-i} denotes the network derived from g by removing all direct links of agent i



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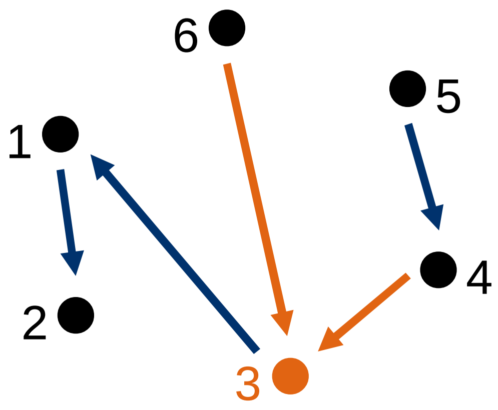

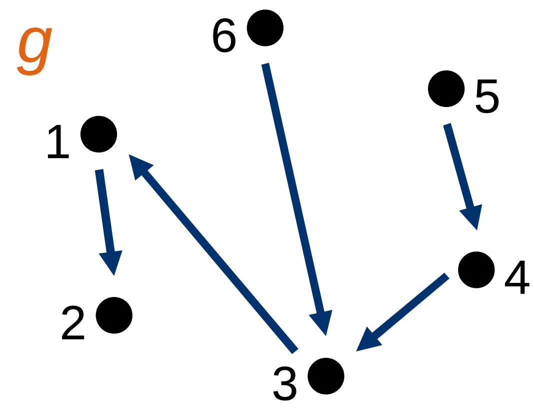

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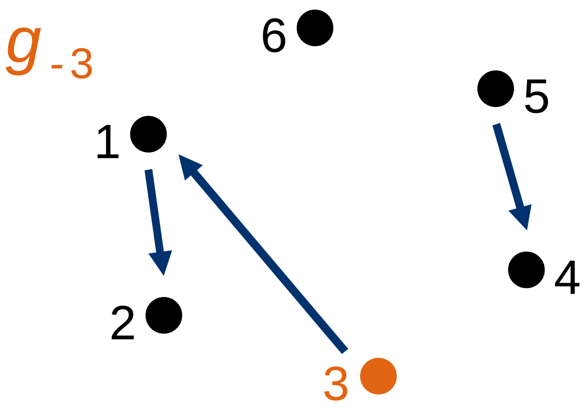

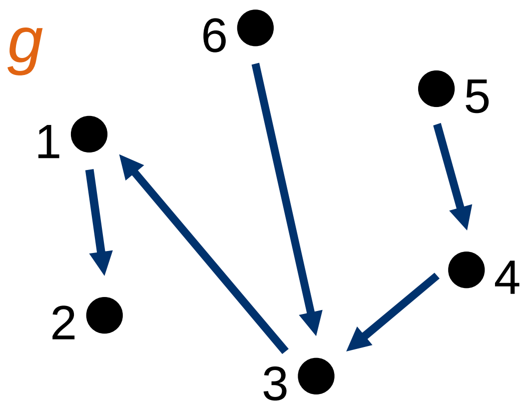

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If $c_{ik} > \sum_{j \neq i} v_{ij}$ for all agents $k \neq i$,

then the only best response for agent i is the empty set Φ

Owner-homogeneous Costs

For each agent *i* all links are equally expensive: $c_{ij} = c_i$ for all *j*

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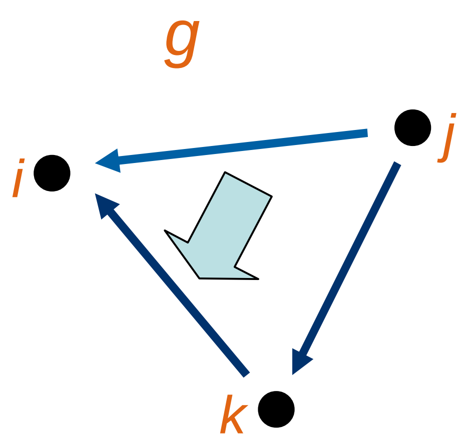
For each agent i all links are equally expensive: $c_{ij} = c_i$ for all j

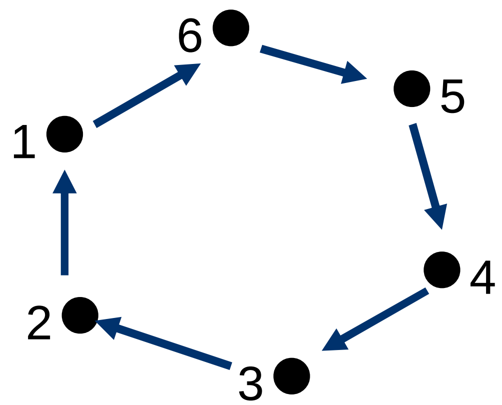
Obs. for owner-homogeneous costs
If link (j,k) exists in g,
then for agent $i \neq j,k$, linking with k
is at least as good as linking with j

"Downstream Efficiency"



Lemma

For any network formation game (N,v,c)
with owner-homogeneous costs and
with $c_i \leq \Sigma_{j \neq i} v_{ij}$ for all agents i,
all *cycle networks* are Nash networks



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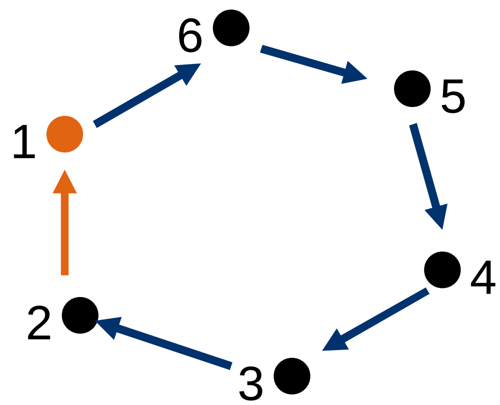

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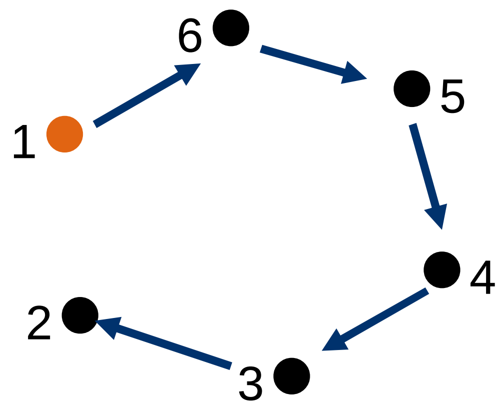
When removing (2,1) agent 1 looses profits from agents 2, 3, 4, 5, 6



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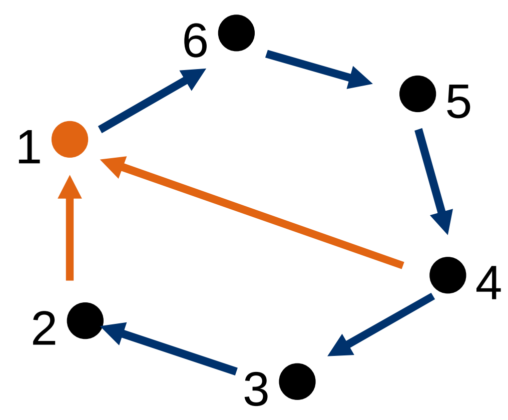
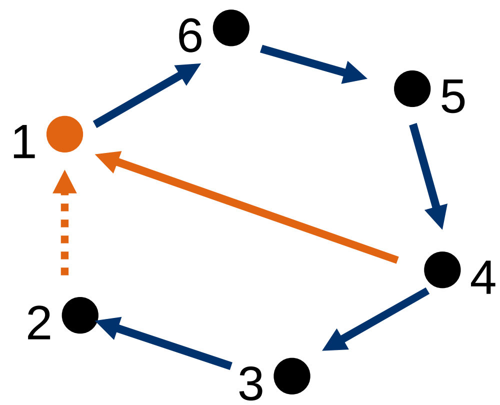
When adding (4,1) agent 1 pays an additional cost of c_{14}



Lemma

For any network formation game (N,v,c) with owner-homogeneous costs and with $c_i \leq \Sigma_{j \neq i}\, v_{ij}$ for all agents i, all *cycle networks* are Nash networks

When replacing (2,1) by (4,1) agent 1 looses profits from agents 2 and 3



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Suppose that (N,v,c) is a network game with n agents
for which NO Nash network exists.

Recall the Lemma:

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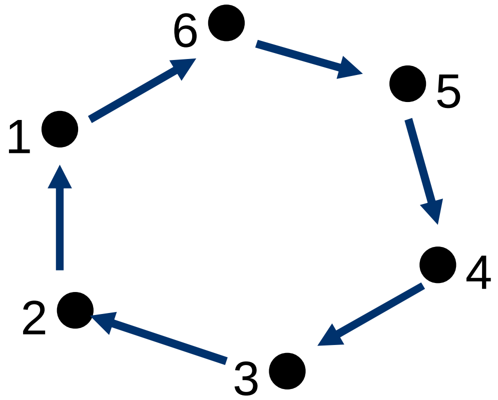

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This agent i can not be agent n

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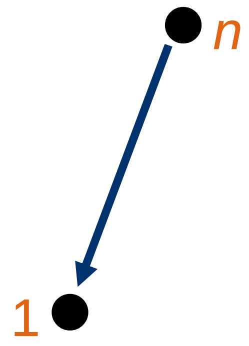
and he has a best response T with $n \in T$

and therefore $c_1 \leq v_{1n}$



Proof continued:

Now recall that for any other agent i
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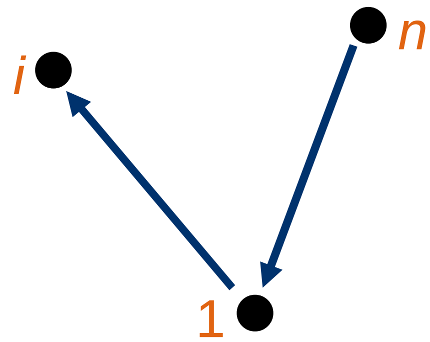
$$\text{Define } v_{ij}^* = \begin{cases} v_{ij} & \text{for } j \neq 1 \\ v_{i1} + v_{in} & \text{for } i \neq 1,\ j = 1 \\ v_{11} + v_{1n} - c_1 & \text{for } i = 1,\ j = 1 \end{cases}$$



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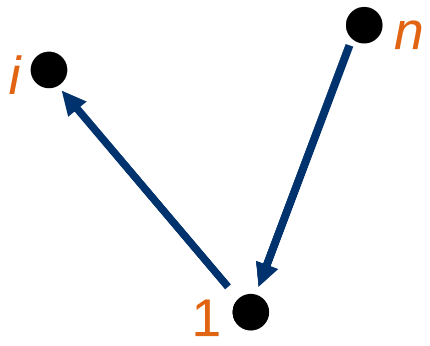

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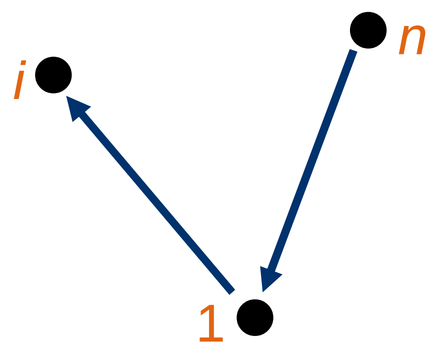

The game (N', v^*, c') has a Nash network g^*

Proof continued:

This network g^* can not be a Nash network in (N,v,c)
Hence at least one agent is not playing a best response

However, it can not be agent n
and any other agent improving in (N,v,c)
contradicts that g^* is a Nash network in (N',v^*,c')
by the way that v^* and v are related to eachother

Example

For network formation games (N,v,c)
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Nash networks do not need to exist

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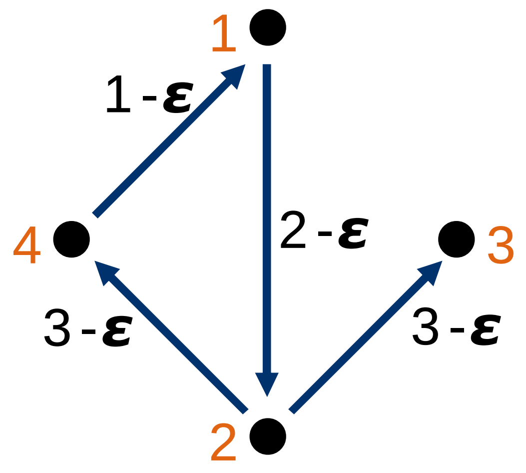
A heterogeneous costs structure

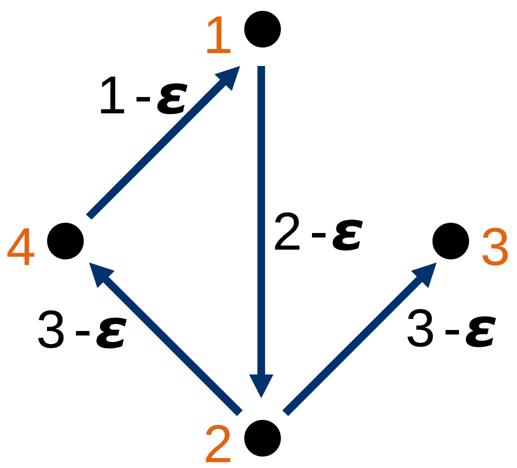
other links to agent 1 cost $1+\varepsilon$
other links to agent 2 cost $2+\varepsilon$
other links to agents 3 and 4 cost $3+\varepsilon$

profits $v_{ij}=1$ for all i and j



Example Explained

The cost/payoff structure



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The arguments (part A)

In any Nash network
agent 3 and agent 4
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Example Explained

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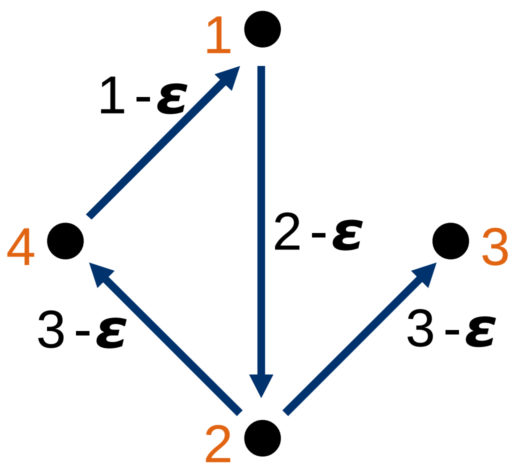

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In any Nash network agent 3 and agent 4 would either play {2} or Φ.

If agent 4 plays {2}, then agent 1 plays {4}.

Example Explained

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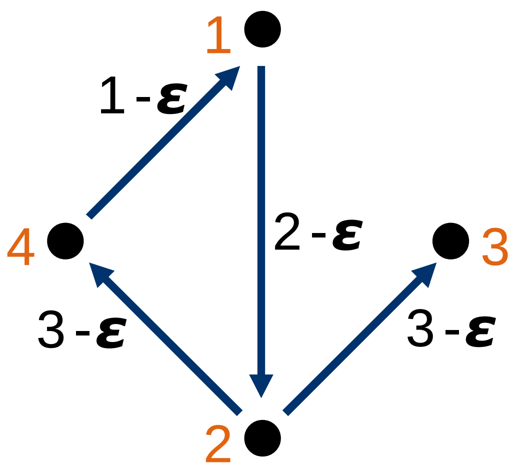

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The arguments (part A)

In any Nash network
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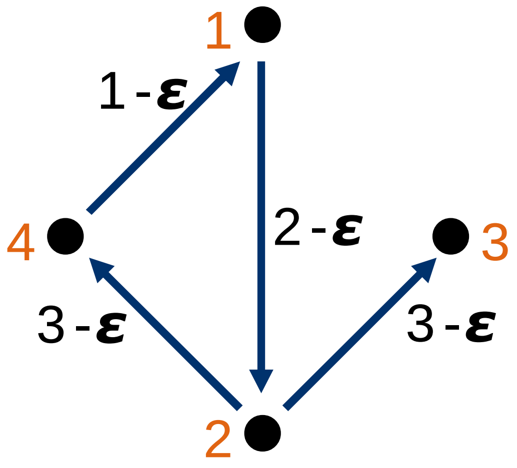
If agent 4 plays {2},
then agent 1 plays {4}.
Then agent 2 plays {1},
because agent 3 never plays {1}.

Example Explained

The cost/payoff structure



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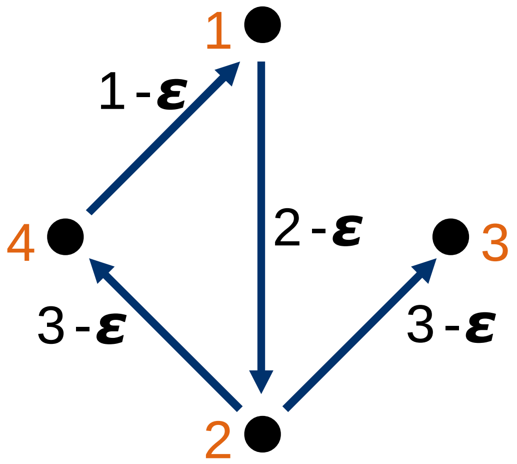

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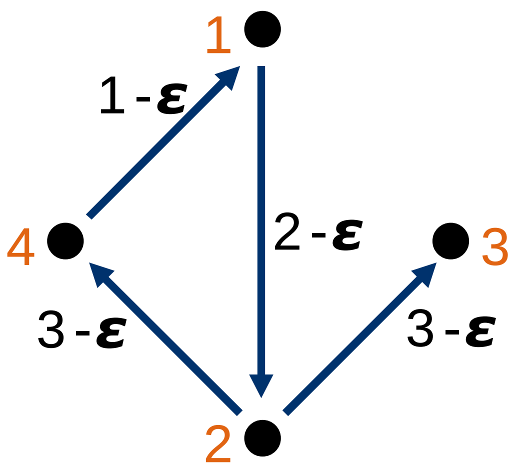
Then agent 3 plays {2}.
Then agent 4 should play Φ.

Example Explained

The cost/payoff structure



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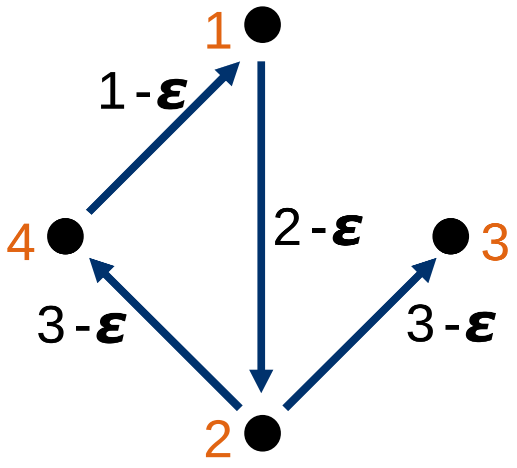
A contradiction

Example Explained

The cost/payoff structure



other links to 1 cost $1+\varepsilon$

other links to 2 cost $2+\varepsilon$

other links to 3, 4 cost $3+\varepsilon$

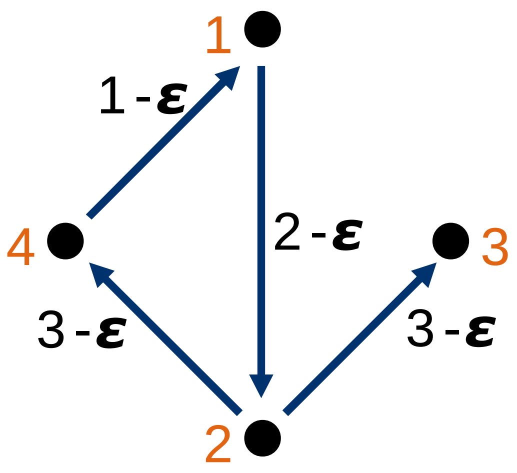
profits $v_{ij}=1$ for all i and j

The arguments (part B)

In any Nash network agent 3 and agent 4 would either play {2} or Φ.
If agent 4 plays Φ, then agent 1 plays S containing 4.

Example Explained

The cost/payoff structure



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The arguments (part B)

In any Nash network
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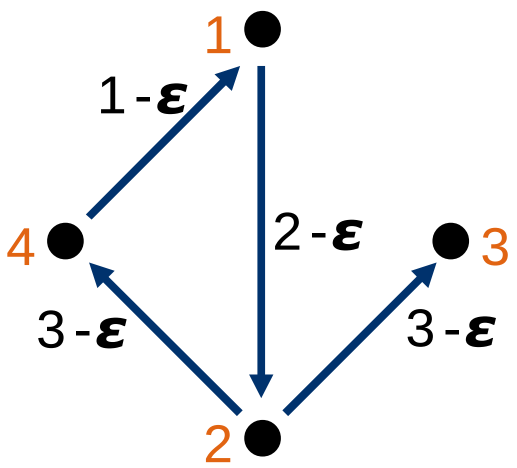
If agent 4 plays Φ,
then agent 1 plays S containing 4.
Then agent 2 plays {1}.

Example Explained

The cost/payoff structure



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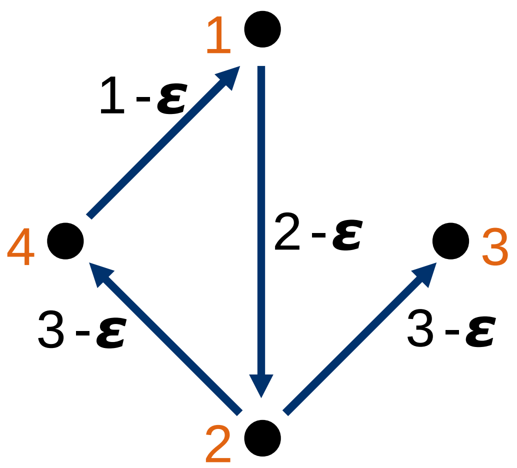
The arguments (part B)

In any Nash network
agent 3 and agent 4
would either play {2} or Φ.
If agent 4 plays Φ,
then agent 1 plays S containing 4.
Then agent 2 plays {1}.
Then agent 3 plays {2}.

Example Explained

The cost/payoff structure



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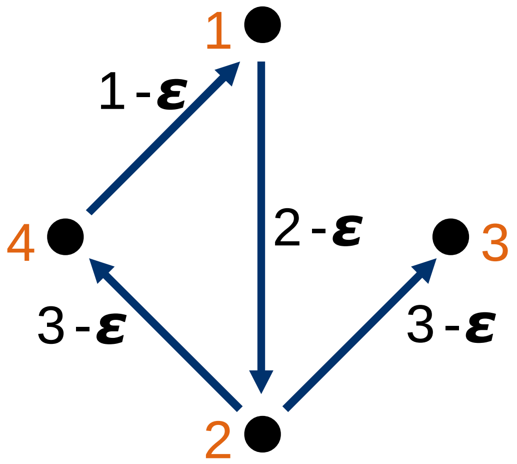
profits v_{ij} =1 for all i and j

The arguments (part B)

In any Nash network
agent 3 and agent 4
would either play {2} or **Φ**.
If agent 4 plays **Φ**,
then agent 1 plays S containing 4.
Then agent 2 plays {1}.
Then agent 3 plays {2}.
Then agent 1 plays {3,4}.

Example Explained

The cost/payoff structure



other links to 1 cost 1+ε

other links to 2 cost 2+ε

other links to 3, 4 cost 3+ε

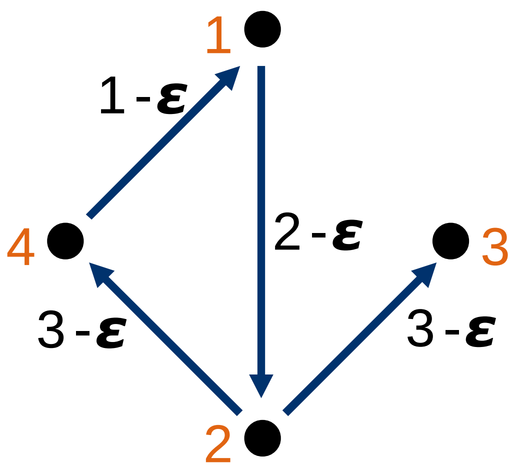
profits v_{ij} =1 for all i and j

The arguments (part B)

In any Nash network
agent 3 and agent 4
would either play {2} or Φ.
If agent 4 plays Φ,
then agent 1 plays S containing 4.
Then agent 2 plays {1}.
Then agent 3 plays {2}.
Then agent 1 plays {3,4}.
Then agent 4 should play {2}.

Example Explained

The cost/payoff structure



other links to 1 cost 1+ε

other links to 2 cost 2+ε

other links to 3, 4 cost 3+ε

profits v_{ij} =1 for all i and j

The arguments (part B)

In any Nash network
agent 3 and agent 4
would either play {2} or **Φ**.
If agent 4 plays **Φ**,
then agent 1 plays *S* containing 4.
Then agent 2 plays {1}.
Then agent 3 plays {2}.
Then agent 1 plays {3,4}.
Then agent 4 should play {2}.

Again a contradiction

Concluding remarks

Our proof implies that for owner-homogeneous costs Nash networks exist that contain at most one cycle and where every vertex has outdegree at most 1

Concluding remarks

Our proof implies that for the owner-homogeneous costs case Nash networks exist that contain at most one cycle and where every vertex has outdegree at most 1

Our model is based mainly on:

- V. Bala & S. Goyal (2000): A non-cooperative model of network formation. *Econometrica* 68, 1181-1229.
- A. Galeotti (2006): One-way flow networks: the role of heterogeneity. *Economic Theory* 29, 163-179.

Independently, an alternative proof for our theorem is given by:

- P. Billand, C. Bravard, S. Sarangi (2008): Existence of Nash Networks in one-way flow models. *Economic Theory.*

Time for questions

a preprint is available at my homepage

comments are welcome at frank@micc.unimaas.nl

Thank you for your attention!

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