One-Way Flow Nash Networks

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Joint work with
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Outline

- **The Model of One-Way Flow Networks**
- **An Existence Result**
- **A Counterexample**
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- A Counterexample
The Model of One-Way Flow Networks

Network Formation Game \((N,v,c)\)
- \(N=\{1,2,3,...,n\}\)
- \(v_{ij}\) is the profit for agent \(i\) for being connected to agent \(j\)
- \(c_{ij}\) is the cost for agent \(i\) for making a link to agent \(j\)
The Model of One-Way Flow Networks

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Example of a network \(g\)
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Example of a network \(g\)

Agent 1 is connected to agents 3, 4, 5 and 6 and obtains profits \(v_{13}, v_{14}, v_{15}, v_{16}\)
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Example of a network \(g\)

Agent 1 is connected to agents 3, 4, 5 and 6 and obtains profits \(v_{13}, v_{14}, v_{15}, v_{16}\)
Agent 1 is not connected to agent 2
Agent 1 has to pay \(c_{13}\) for the link \((3,1)\)
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- \(v_{ij}\) is the profit for agent \(i\) for being connected to agent \(j\)
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Example of a network \(g\)

The payoff \(n_1(g)\) for agent 1 is

\[
 n_1(g) = v_{13} + v_{14} + v_{15} + v_{16} - c_{13}
\]
The Model of One-Way Flow Networks

More generally

\[ n_i(g) = \sum_{j \in N_i(g)} v_{ij} - \sum_{j \in N_{di}(g)} c_{ij} \]

where \( N_i(g) \) is the set of agents that \( i \) is connected to, and where \( N_{di}(g) \) is the set of agents that \( i \) is directly connected to.
The Model of One-Way Flow Networks

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An action for agent \( i \) is any subset \( S \) of \( N \setminus \{i\} \) indicating the set of agents that \( i \) connects to directly.
The Model of One-Way Flow Networks

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\[ n_i(g) = \sum_{j \in N_i(g)} v_{ij} - \sum_{j \in Nd_i(g)} c_{ij} \]

where \( N_i(g) \) is the set of agents that \( i \) is connected to, and where \( Nd_i(g) \) is the set of agents that \( i \) is directly connected to.

An action for agent \( i \) is any subset \( S \) of \( N \setminus \{i\} \) indicating the set of agents that \( i \) connects to directly.

A network \( g \) is a Nash network if each agent \( i \) is playing a best response in terms of his individual payoff \( n_i(g) \).
A Closer Look at Nash Networks

A network $g$ is a Nash network if for each agent $i$

$$\pi_i(g) \geq \pi_i(g_{-i} + \{(j,i) : j \in S\})$$

for all subsets $S$ of $N \setminus \{i\}$

Here $g_{-i}$ denotes the network derived from $g$ by removing all direct links of agent $i$
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A set $S$ that maximizes the right-hand side of above expression is called a **best response** for agent $i$ to the network $g$

In a Nash network all agents are linked to their best responses
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In a Nash network all agents are linked to their best responses

If $c_{ik} > \Sigma_{j\neq i} v_{ij}$ for all agents $k \neq i$,

then the only best response for agent $i$ is the empty set $\emptyset$
Owner-homogeneous Costs

For each agent $i$ all links are equally expensive: $c_{ij} = c_i$ for all $j$
Owner-homogeneous Costs

For each agent $i$ all links are equally expensive: $c_{ij} = c_i$ for all $j$

Obs. for owner-homogeneous costs
If link $(j,k)$ exists in $g$,
then for agent $i \neq j,k$, linking with $k$
is at least as good as linking with $j$

“Downstream Efficiency”
Lemma

For any network formation game \((N, v, c)\) with owner-homogeneous costs and with \(c_i \leq \Sigma_{j \neq i} v_{ij}\) for all agents \(i\), all cycle networks are Nash networks.
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When removing \((2,1)\) agent 1 looses profits from agents 2, 3, 4, 5, 6.
Lemma

For any network formation game \((N, v, c)\) with owner-homogeneous costs and with \(c_i \leq \sum_{j \neq i} v_{ij}\) for all agents \(i\), all cycle networks are Nash networks.

When adding \((4,1)\) agent 1 pays an additional cost of \(c_{14}\).
Lemma

For any network formation game \((N,v,c)\) with owner-homogeneous costs and with \(c_i \leq \Sigma_{j \neq i} v_{ij}\) for all agents \(i\), all cycle networks are Nash networks.

When replacing \((2,1)\) by \((4,1)\) agent 1 looses profits from agents 2 and 3.
Theorem

For any network formation game \((N,v,c)\) with owner-homogeneous costs, a Nash network exists
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Proof by induction to the number of agents:
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Induction hypothesis: Nash networks exist for all network games with less than \(n\) agents.
Theorem

For any network formation game \((N,\nu,c)\) with owner-homogeneous costs, a Nash network exists

Proof by induction to the number of agents:

If \(n=1\), then the trivial network is a Nash network

Induction hypothesis: Nash networks exist for all network games with less than \(n\) agents.

Suppose that \((N,\nu,c)\) is a network game with \(n\) agents for which NO Nash network exists.
Recall the Lemma:

For any network formation game \((N, v, c)\) with owner-homogeneous costs and with \(c_i \leq \sum_{j \neq i} v_{ij}\) for all agents \(i\), all cycle networks are Nash networks.
Proof continued:

Hence there is at least one agent $i$ with $c_i > \sum_{j \neq i} v_{ij}$
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W.l.o.g. this agent is agent $n$
Proof continued:

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W.l.o.g. this agent is agent $n$ 
Consider $(N', v', c')$ with $N' = N \setminus \{n\}$ and with $v$ and $c$ restricted to agents in $N'$
Proof continued:

Hence there is at least one agent $i$ with $c_i > \Sigma_{j\neq i} v_{ij}$

W.l.o.g. this agent is agent $n$

Consider $(N',v',c')$ with $N'=N\setminus\{n\}$

and with $v$ and $c$ restricted to agents in $N'$

Let $g'$ be a Nash network in $(N',v',c')$ (induction hypothesis)
Proof continued:

Hence there is at least one agent $i$ with $c_i > \sum_{j \neq i} v_{ij}$
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Consider $(N', v', c')$ with $N' = N \setminus \{n\}$
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Let $g'$ be a Nash network in $(N', v', c')$ (induction hypothesis)
Then by assumption $g'$ is no Nash network in $(N, v, c)$
Proof continued:

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Then by assumption $g'$ is no Nash network in $(N,v,c)$
Therefore there is an agent $i$
for whom the links in $g$ are no best response
Proof continued:

Hence there is at least one agent $i$ with $c_i > \Sigma_{j\neq i} v_{ij}$
W.l.o.g. this agent is agent $n$
Consider $(N',v',c')$ with $N'=N\{n\}$
and with $v$ and $c$ restricted to agents in $N'$
Let $g'$ be a Nash network in $(N',v',c')$ (induction hypothesis)
Then by assumption $g'$ is no Nash network in $(N,v,c)$
Therefore there is an agent $i$ for whom the links in $g$ are no best response
This agent $i$ can not be agent $n$
W.l.o.g. this agent is agent 1
and he has a best response $T$ with $n \in T$
and therefore $c_1 \leq v_{1n}$
Proof continued:

Now recall that for any other agent $i$
linking to agent 1 would be at least as good as linking to agent $n$

Define $v_{ij}^* = \begin{cases} 
v_{ij} & \text{for } j \neq 1 \\
v_{i1} + v_{in} & \text{for } i \neq 1, j = 1 \\
v_{11} + v_{1n} - c_1 & \text{for } i = 1, j = 1 \end{cases}$
Proof continued:

Now recall that for any other agent $i$
linking to agent 1 would be at least as good as linking to agent $n$

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Now $n^*_i(g) = n_i(g + (n,1))$
for any network $g$ on $N'$
and for any agent $i$ in $N'$
Proof continued:

Now recall that for any other agent $i$ linking to agent $1$ would be at least as good as linking to agent $n$.

Define $v_{ij}^* = \begin{cases} v_{ij} & \text{for } j \neq 1 \\ v_{i1} + v_{in} & \text{for } i \neq 1, j = 1 \\ v_{11} + v_{1n} - c_1 & \text{for } i = 1, j = 1 \end{cases}$

Now $n_i^*(g) = n_i(g + (n,1))$ for any network $g$ on $N'$ and for any agent $i$ in $N'$.

The game $(N', v^*, c')$ has a Nash network $g^*$. 
Proof continued:

This network $g^*$ can not be a Nash network in $(N,v,c)$
Hence at least one agent is not playing a best response

However, it can not be agent $n$
and any other agent improving in $(N,v,c)$
contradicts that $g^*$ is a Nash network in $(N',v^*,c')$
by the way that $v^*$ and $v$ are related to each other
Example

For network formation games \((N,v,c)\) with heterogeneous costs, Nash networks do not need to exist.
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A heterogeneous costs structure

other links to agent 1 cost \(1 + \epsilon\)
other links to agent 2 cost \(2 + \epsilon\)
other links to agents 3 and 4 cost \(3 + \epsilon\)

profits \(v_{ij} = 1\) for all \(i\) and \(j\)
Example Explained

The cost/payoff structure

The arguments (part A)

In any Nash network agent 3 and agent 4 would either play \{2\} or \emptyset.

other links to 1 cost 1+ε
other links to 2 cost 2+ε
other links to 3, 4 cost 3+ε
profits \(v_{ij} = 1\) for all \(i\) and \(j\)
Example Explained

The cost/payoff structure

- Other links to 1 cost 1+ε
- Other links to 2 cost 2+ε
- Other links to 3, 4 cost 3+ε
- Profits $v_{ij} = 1$ for all $i$ and $j$

The arguments (part A)

In any Nash network agent 3 and agent 4 would either play \{2\} or $\emptyset$. If agent 4 plays \{2\}, then agent 1 plays \{4\}.
Example Explained

The cost/payoff structure

- Other links to 1 cost $1+\varepsilon$
- Other links to 2 cost $2+\varepsilon$
- Other links to 3, 4 cost $3+\varepsilon$
- Profits $v_{ij} = 1$ for all $i$ and $j$

The arguments (part A)

In any Nash network, agent 3 and agent 4 would either play $\{2\}$ or $\emptyset$. If agent 4 plays $\{2\}$, then agent 1 plays $\{4\}$. Then agent 2 plays $\{1\}$, because agent 3 never plays $\{1\}$. 
Example Explained

The cost/payoff structure

other links to 1 cost $1+\epsilon$
other links to 2 cost $2+\epsilon$
other links to 3, 4 cost $3+\epsilon$
profits $v_{ij} = 1$ for all $i$ and $j$

The arguments (part A)

In any Nash network agent 3 and agent 4 would either play $\{2\}$ or $\emptyset$. If agent 4 plays $\{2\}$, then agent 1 plays $\{4\}$. Then agent 2 plays $\{1\}$, because agent 3 never plays $\{1\}$. Then agent 3 plays $\{2\}$. 
Example Explained

The cost/payoff structure

- Other links to 1 cost 1 + ε
- Other links to 2 cost 2 + ε
- Other links to 3, 4 cost 3 + ε
- Profits $v_{ij} = 1$ for all $i$ and $j$

The arguments (part A)

In any Nash network, agent 3 and agent 4 would either play \{2\} or $\Phi$. If agent 4 plays \{2\}, then agent 1 plays \{4\}. Then agent 2 plays \{1\}, because agent 3 never plays \{1\}. Then agent 3 plays \{2\}. Then agent 4 should play $\Phi$. 
Example Explained

The cost/payoff structure

1

1-\(\varepsilon\)

4

3-\(\varepsilon\)

2

2-\(\varepsilon\)

other links to 1 cost 1+\(\varepsilon\)
other links to 2 cost 2+\(\varepsilon\)
other links to 3, 4 cost 3+\(\varepsilon\)
profits \(v_{ij} = 1\) for all \(i\) and \(j\)

The arguments (part A)

In any Nash network agent 3 and agent 4 would either play \{2\} or \(\Phi\).

If agent 4 plays \{2\}, then agent 1 plays \{4\}.

Then agent 2 plays \{1\}, because agent 3 never plays \{1\}.

Then agent 3 plays \{2\}.

Then agent 4 should play \(\Phi\).

A contradiction
Example Explained

The cost/payoff structure

other links to 1 cost $1+\varepsilon$
other links to 2 cost $2+\varepsilon$
other links to 3, 4 cost $3+\varepsilon$
profits $v_{ij} = 1$ for all $i$ and $j$

The arguments (part B)

In any Nash network
agent 3 and agent 4
would either play $\{2\}$ or $\Phi$.
If agent 4 plays $\Phi$,
then agent 1 plays $S$ containing 4.
Example Explained

The cost/payoff structure

- Other links to 1 cost $1+\varepsilon$
- Other links to 2 cost $2+\varepsilon$
- Other links to 3, 4 cost $3+\varepsilon$
- Profits $v_{ij} = 1$ for all $i$ and $j$

The arguments (part B)

In any Nash network, agent 3 and agent 4 would either play $\{2\}$ or $\Phi$. If agent 4 plays $\Phi$, then agent 1 plays $S$ containing 4. Then agent 2 plays $\{1\}$.
Example Explained

The cost/payoff structure

- Other links to 1 cost $1+\varepsilon$
- Other links to 2 cost $2+\varepsilon$
- Other links to 3, 4 cost $3+\varepsilon$
- Profits $v_{ij} = 1$ for all $i$ and $j$

The arguments (part B)

In any Nash network agent 3 and agent 4 would either play $\{2\}$ or $\emptyset$.
If agent 4 plays $\emptyset$, then agent 1 plays $S$ containing 4.
Then agent 2 plays $\{1\}$.
Then agent 3 plays $\{2\}$. 
Example Explained

The cost/payoff structure

- Other links to 1 cost $1+\varepsilon$
- Other links to 2 cost $2+\varepsilon$
- Other links to 3, 4 cost $3+\varepsilon$
- Profits $v_{ij} = 1$ for all $i$ and $j$

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In any Nash network, agent 3 and agent 4 would either play $\{2\}$ or $\Phi$. If agent 4 plays $\Phi$, then agent 1 plays $S$ containing 4. Then agent 2 plays $\{1\}$. Then agent 3 plays $\{2\}$. Then agent 1 plays $\{3,4\}$. 
Example Explained

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other links to 1 cost $1+\varepsilon$
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The arguments (part B)

In any Nash network agent 3 and agent 4 would either play \{2\} or $\Phi$. If agent 4 plays $\Phi$, then agent 1 plays $S$ containing 4. Then agent 2 plays \{1\}. Then agent 3 plays \{2\}. Then agent 1 plays \{3,4\}. Then agent 4 should play \{2\}. 
Example Explained

The cost/payoff structure

other links to 1 cost \( 1 + \varepsilon \)
other links to 2 cost \( 2 + \varepsilon \)
other links to 3, 4 cost \( 3 + \varepsilon \)
profits \( v_{ij} = 1 \) for all \( i \) and \( j \)

The arguments (part B)

In any Nash network
agent 3 and agent 4
would either play \( \{2\} \) or \( \Phi \).
If agent 4 plays \( \Phi \),
then agent 1 plays \( S \) containing 4.
Then agent 2 plays \( \{1\} \).
Then agent 3 plays \( \{2\} \).
Then agent 1 plays \( \{3,4\} \).
Then agent 4 should play \( \{2\} \).

Again a contradiction
Concluding remarks

Our proof implies that for owner-homogeneous costs Nash networks exist that contain at most one cycle and where every vertex has outdegree at most 1.
Concluding remarks

Our proof implies that for the owner-homogeneous costs case Nash networks exist that contain at most one cycle and where every vertex has outdegree at most 1.

Our model is based mainly on:

Independently, an alternative proof for our theorem is given by:
Time for questions

a preprint is available at my homepage

comments are welcome at frank@micc.unimaas.nl
Thank you for your attention!

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