Maastricht University *Department of Knowledge Engineering*

One-Way Flow Nash Networks

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Outline

- The Model of One-Way Flow Networks
- An Existence Result and a Structural Observation
- A Dynamic Procedure of Local Actions
- A Counterexample

The Model of One-Way Flow Networks

Network Formation Game (N,v,c)

- $N = \{1,2,3,\ldots,n\}$
- $v_{ij} \geq 0$ is the profit for agent i for being connected to agent j
- $c_{ij} \geq 0$ is the cost for agent i for being *directly* connected to agent j

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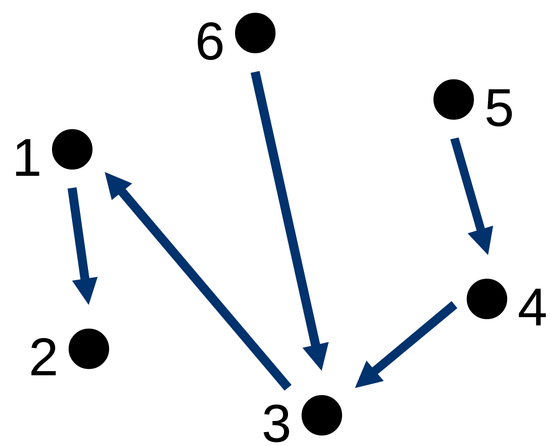
Example of a one-way flow network g



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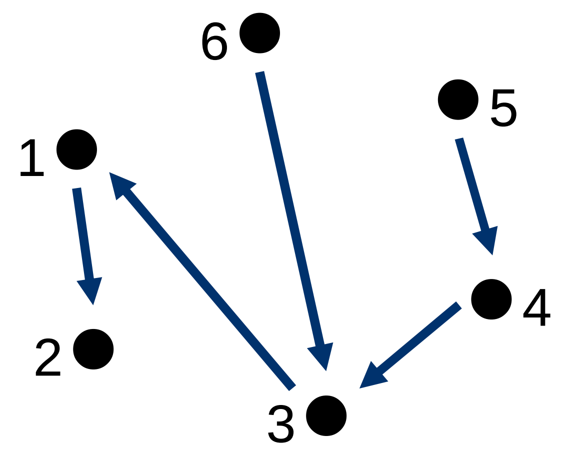
Example of a one-way flow network g

Agent 1 is connected to agents 3,4,5 and 6 and obtains profits v_{13} , v_{13} , v_{14} , v_{15} , v_{16} .
Agent 1 is *not* connected to agent 2.
Agent 1 has to pay c_{13} for the link (3,1).



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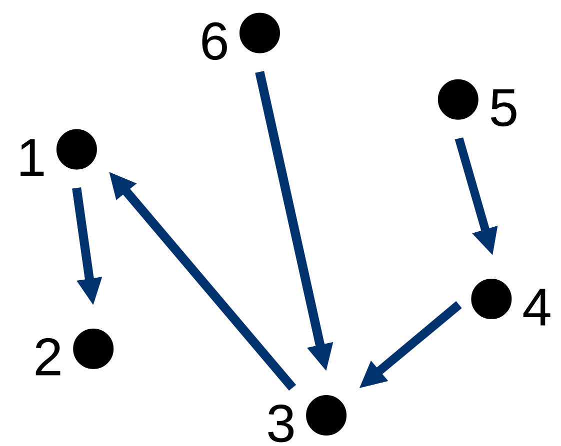

The payoff $\pi_1(g)$ for agent 1 is

$$\pi_1(g) = v_{13} + v_{14} + v_{15} + v_{16} - c_{13}.$$

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More generally:

$$\pi_i(g) = \Sigma_{j \in Ni(g)} v_{ij} - \Sigma_{j \in Ndi(g)} c_{ij}$$

where $N_i(g)$ is the set of agents that i is connected to in g, and
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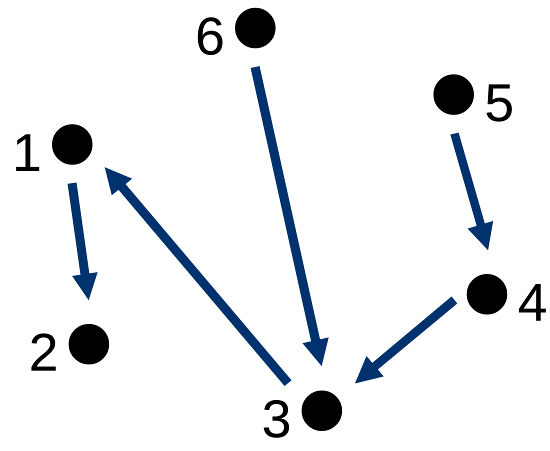
Our model is mainly based on:

- V. Bala & S. Goyal (2000): A non-cooperative model of network formation. *Econometrica* 68, 1181-1229.
- A. Galeotti (2006): One-way flow networks: the role of heterogeneity. *Economic Theory* 29, 163-179.

The Model of One-Way Flow Networks

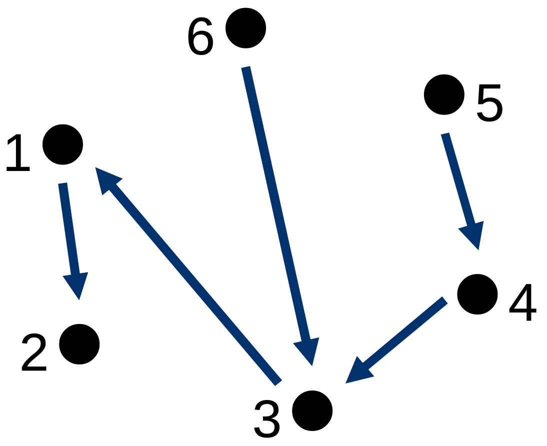
An action for agent i is any subset S of $N \setminus \{i\}$ indicating the set of agents that i connects to directly.



The Model of One-Way Flow Networks

An action for agent i is any subset S of $N \setminus \{i\}$ indicating the set of agents that i connects to directly.

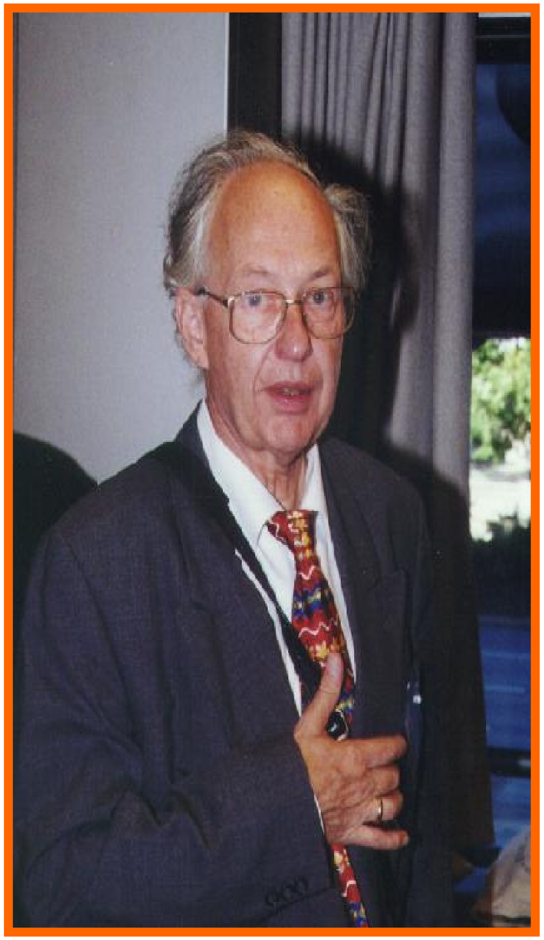
A network g is a Nash network if each agent i is playing a best response in terms of his individual payoff $\pi_i(g)$.



“A Beautiful Mind ”

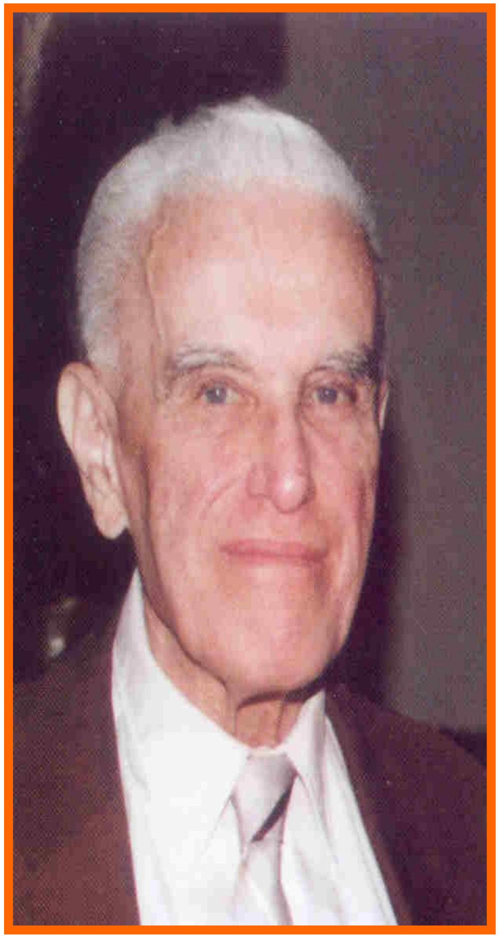
John F. Nash

Non-cooperative games, *Annals of Mathematics* 54, 1951



Reinhard Selten

John F. Nash



John C. Harsanyi

1994: Nobel Prize for Economics

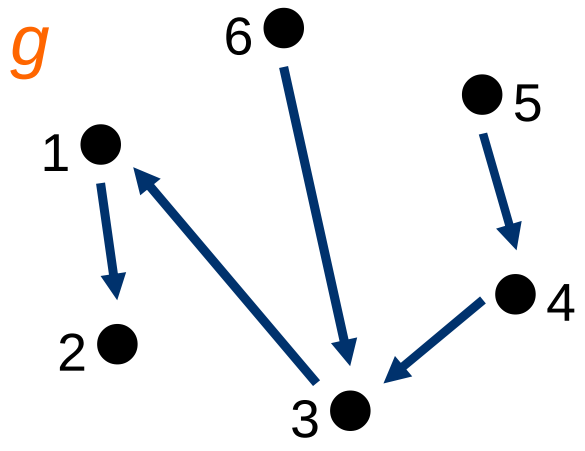
A Closer Look at Nash Networks

A network g is a Nash network if for each agent i

$$\pi_i(g) \geq \pi_i(g_{-i} + \{(j,i) : j \in S\})$$

for all subsets S of $N \setminus \{i\}$.

Here g_{-i} denotes the network derived from g by removing all direct links of agent i.



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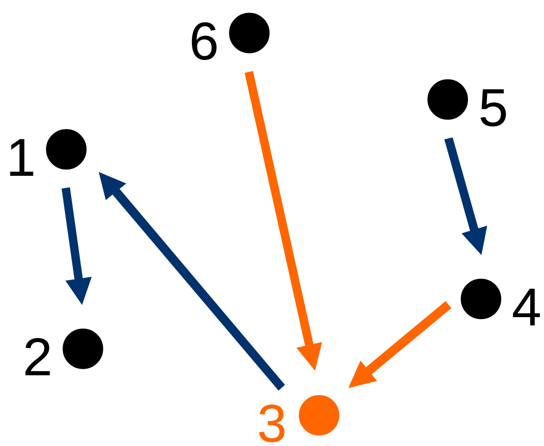

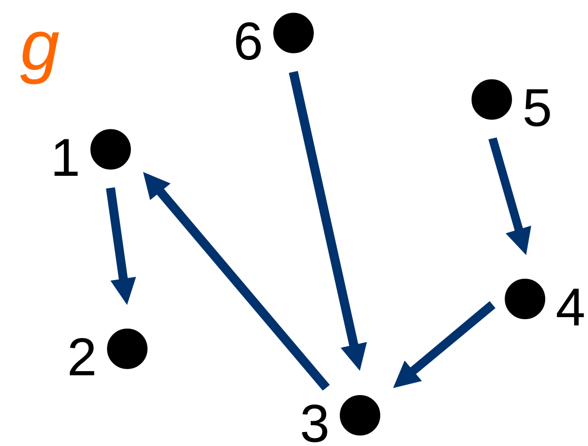

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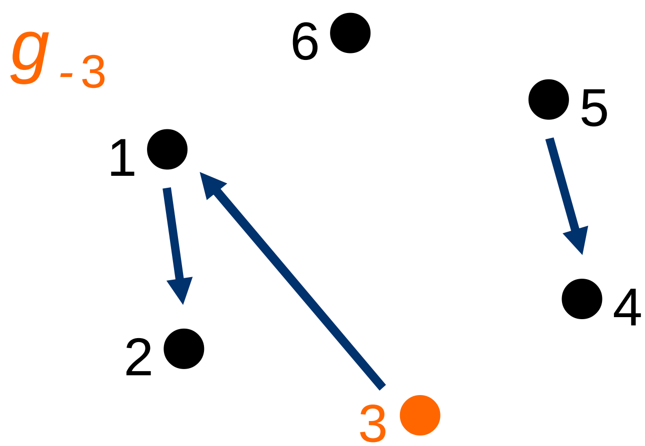

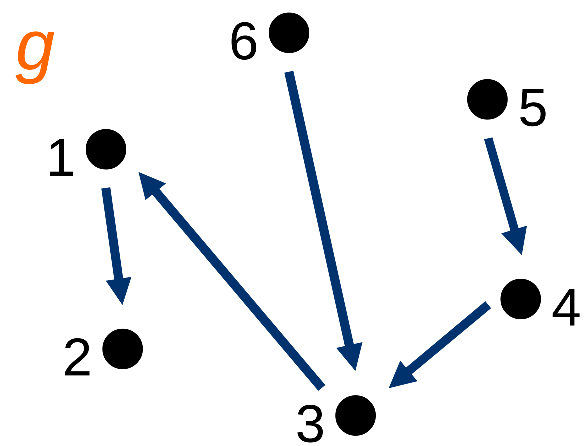

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Remark: If $c_{ik} > \sum_{j \neq i} v_{ij}$ for all agents $k \neq i$, then the only best response for agent i is the empty set Φ.

Owner-Homogeneous Costs

For each agent i all links are equally expensive: $c_{ij} = c_i$ for all j.

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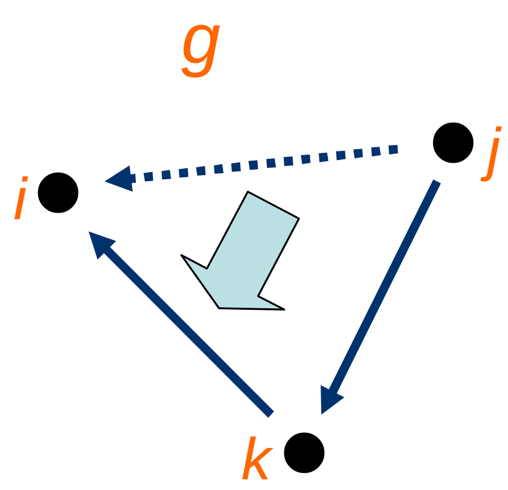
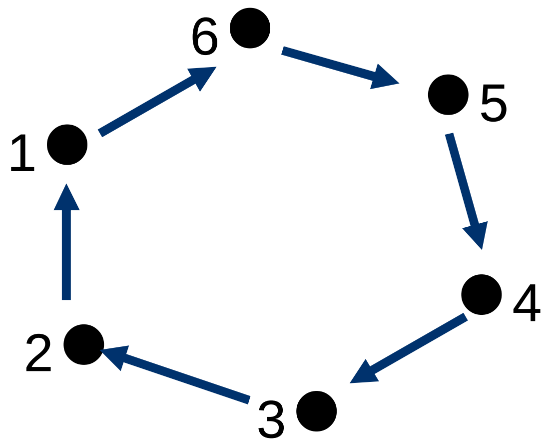
Observation for owner-homogeneous costs
If link (j,k) exists in g,
then for agent $i \neq j,k$, linking with k
is at least as good as linking with j.

"Downstream Efficiency"



Lemma

For any network formation game (N,v,c)
with owner-homogeneous costs and
with $c_i \leq \Sigma_{j\neq i}\ v_{ij}$ for all agents i,
all *cycle networks* are Nash networks.



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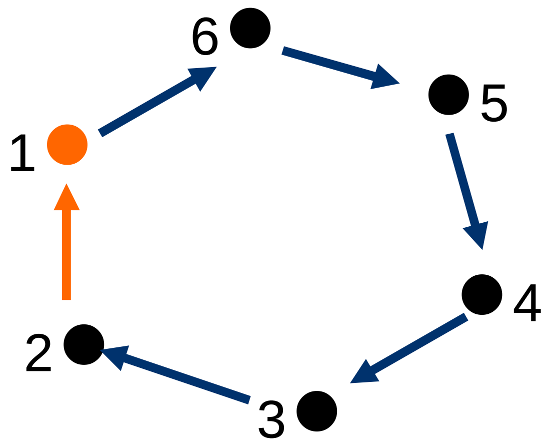

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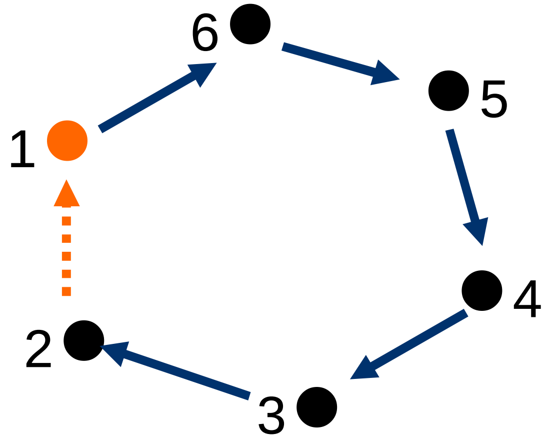
Proof by examining agent 1:

When removing (2,1) agent 1 looses profits from agents 2, 3, 4, 5, 6.



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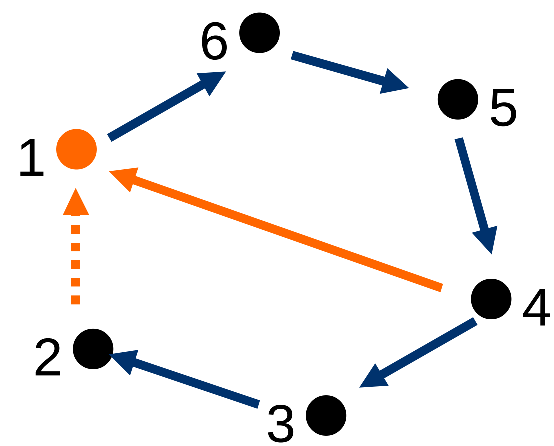
When replacing (2,1) by (4,1) agent 1 looses profits from agents 2 and 3.



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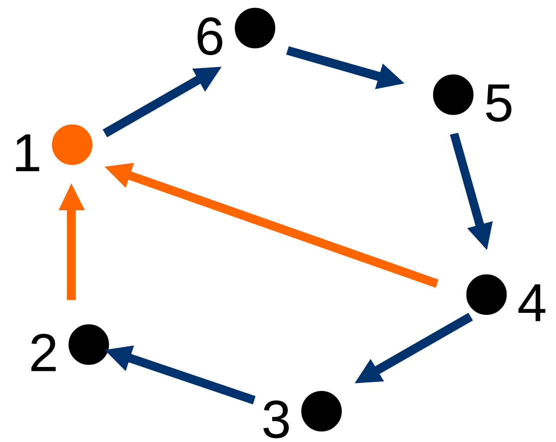
When adding (4,1) agent 1 pays an additional cost of c_{14} .



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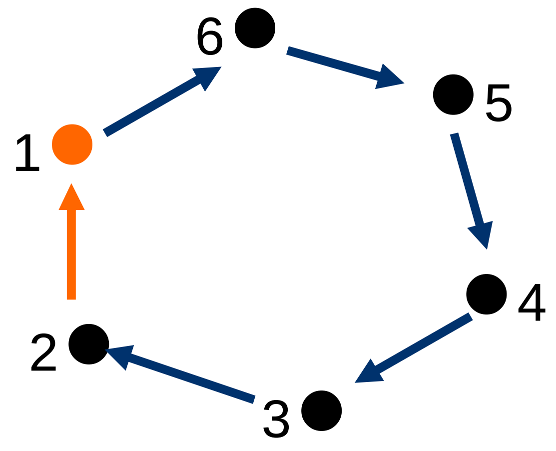
Proof by examining agent 1:

Hence {2} is a best response for agent 1 . ■



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Proof by induction to the number of agents n:

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Induction hypothesis: Nash networks exist for all network games with less than n agents.

Suppose that (N,v,c) is a network game with n agents
for which NO Nash network exists.

Recall the Lemma:

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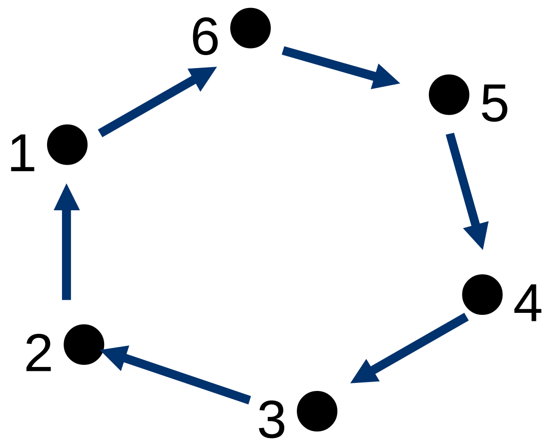

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Consider (N', v', c') with $N' = N \setminus \{n\}$

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Therefore there is an agent i

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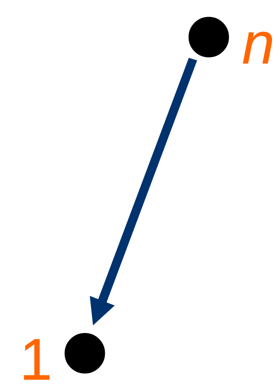
This agent i can not be agent n;

so w.l.o.g. this agent is agent 1

and he has a best response T with $n \in T$

and therefore $c_1 \leq v_{1n}$,

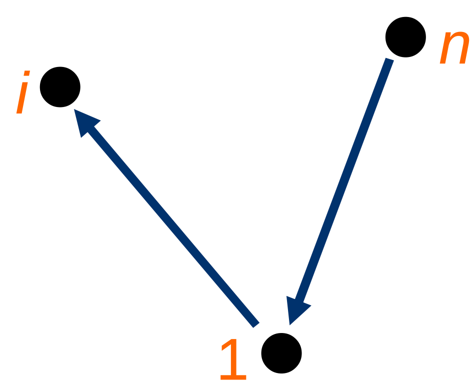
because agent n is not linked to anyone else.



Proof Continued:

Now recall that, by downstream efficiency, for any other agent i linking to agent 1 would be at least as good as linking to agent n.

Define $v_{ij}^* = \begin{cases} v_{ij} & \text{for } j \neq 1 \\ v_{i1} + v_{in} & \text{for } i \neq 1, j = 1 \\ v_{11} + v_{1n} - c_1 & \text{for } i = 1, j = 1 \end{cases}$



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for any network g on N'
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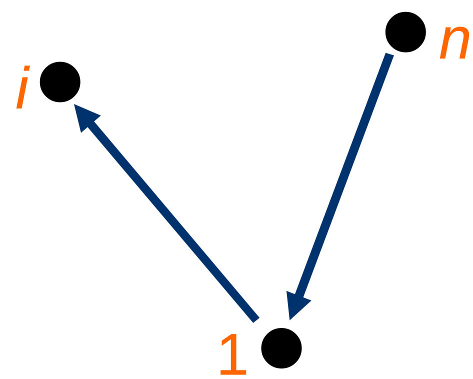

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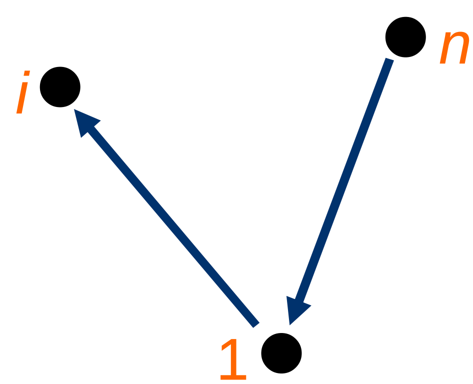
By the induction hypothesis
the game (N', v^*, c') has a Nash network g^*.



Proof Continued:

In g^* all agents in N' play best responses w.r.t. (N', v^*, c') because g^* is a Nash network.

By the way that v^* was defined, this implies that w.r.t. (N,v,c) in g^* all agents in N' play best responses.
And we still have that w.r.t. (N,v,c) the only best response for agent n is to play $\boldsymbol{\Phi}$ in any network, particularly in g^*.

Hence g^* is a Nash network in (N,v,c),
This contradicts the initial assumption
that there is no Nash network in (N,v,c). ■

Observation

For each network formation game with owner-homogeneous costs there exists at least one Nash network with at most one cycle and with every vertex having an out-degree of at most 1...

... but there may well be other Nash networks as well.

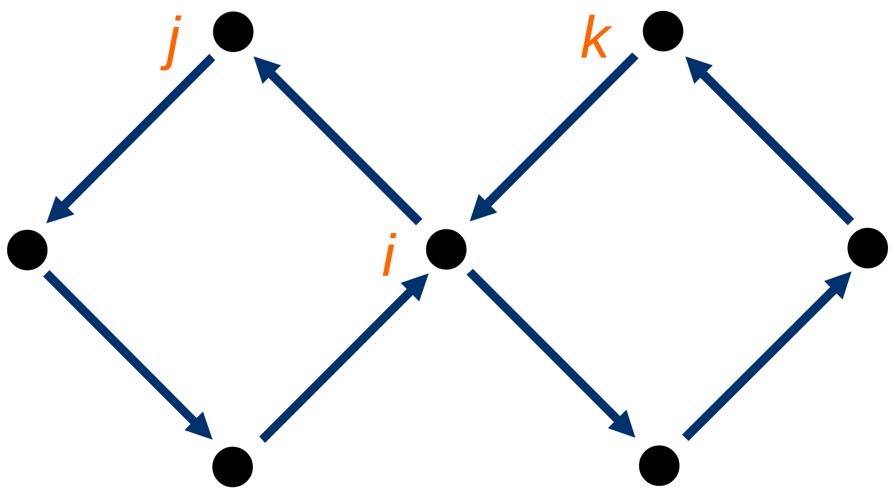
Example

This network g_1 is a Nash network where

$$\pi_i(g_1) = 4, \ \pi_j(g_1) = 5, \ \pi_k(g_1) = 5.$$

Notice that agent i and agent k have only one best response, but agent j is indifferent between linking to i or linking to k.

g_1



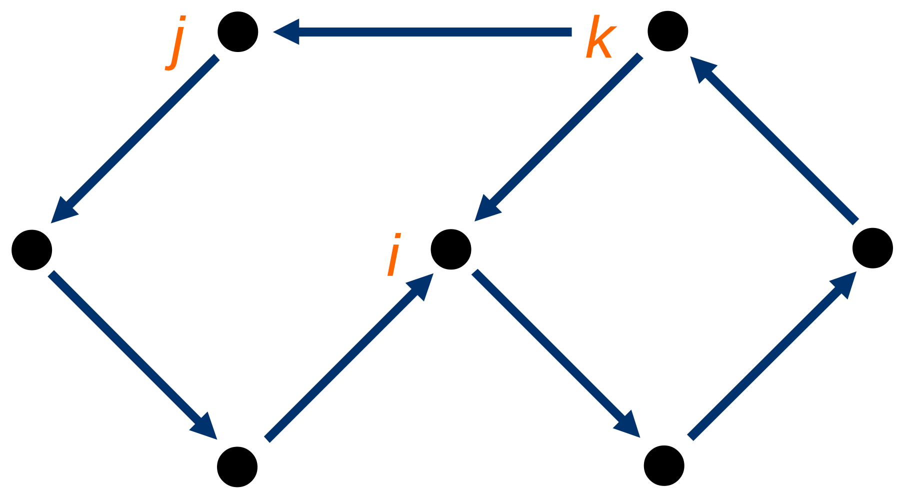
Example

If agent *j* replaces the link to *i* by one to *k*,
then we get the network g_2 where the payoffs are still the same

$\pi_i(g_2) = 4, \ \pi_j(g_2) = 5, \ \pi_k(g_2) = 5.$

However, g_2 is no Nash network since agent *i* can improve his payoff by removing the link to *k*.

g_2

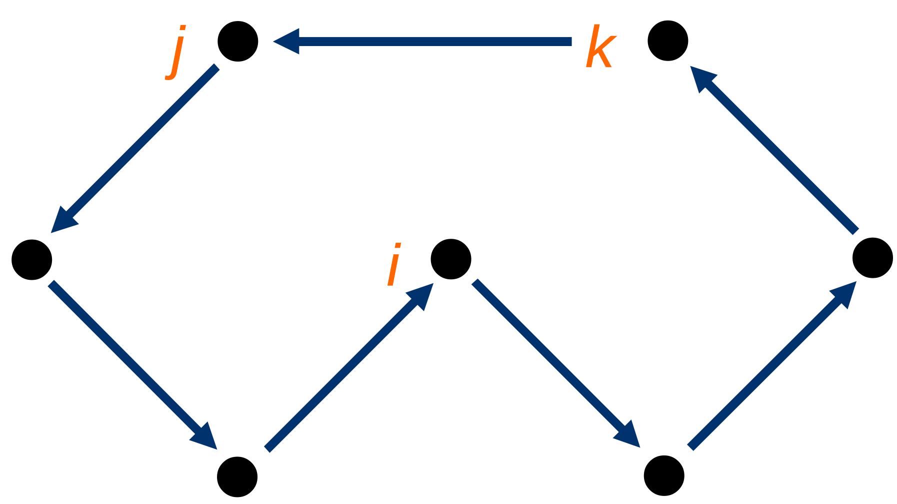


Example

If agent i removes the link to k,
then we get the cycle network g_3 which is a Nash network with

$$\pi_i(g_3) = 5,\ \pi_j(g_3) = 5,\ \pi_k(g_3) = 5.$$

g_3



A Dynamic Procedure of Local Actions

Recall that an action for agent i is any subset S of $N \setminus \{i\}$ indicating the set of agents that i connects to directly.

A local action for agent i in a dynamic context is one of these:

- not changing anything in the network
- deleting one link (j,i)
- adding one link (k,i)
- replacing one link (j,i) by another link (k,i)

A network g is a local Nash network
if each agent i is playing a best local response
in terms of his individual payoff $\pi_i(g)$.

A Dynamic Procedure of Local Actions

Let g_t be the network at stage t and suppose that agent i plays a local action a that leads to the network g_{t+1} then we define action a to be a good local response if

$$\pi_i(g_{t+1}) \geq \pi_i(g_t)$$

A Dynamic Procedure of Local Actions

Start with an arbitrary network g_1 at stage 1.
Let g_t be the network at stage t.
If g_t is a local Nash network with maximum outdegree 1, then stop.
Otherwise, choose an agent i at random
and let i play a random good local response,
which leads to the network g_{t+1} to be examined at stage $t+1$.

Theorem

This procedure ends in a global Nash network
with probability 1.

(Proof skipped here)

Example without Nash Network

For network formation games (N, v, c)
with *heterogeneous* costs,
Nash networks do not need to exist.

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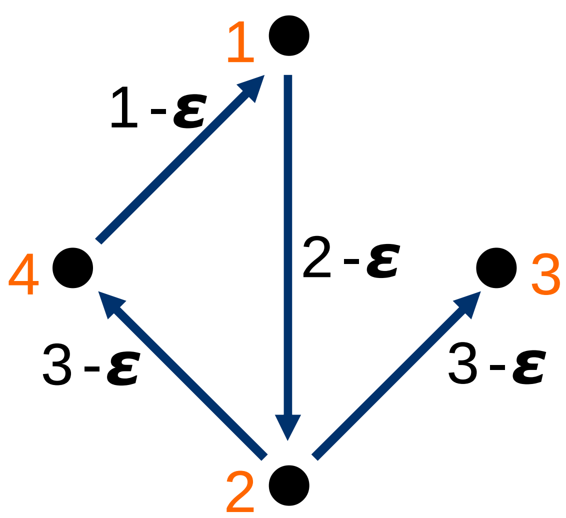
A heterogeneous costs structure

other links to agent 1 cost $1+\varepsilon$
other links to agent 2 cost $2+\varepsilon$
other links to agents 3 and 4 cost $3+\varepsilon$

profits $v_{ij} = 1$ for all i and j



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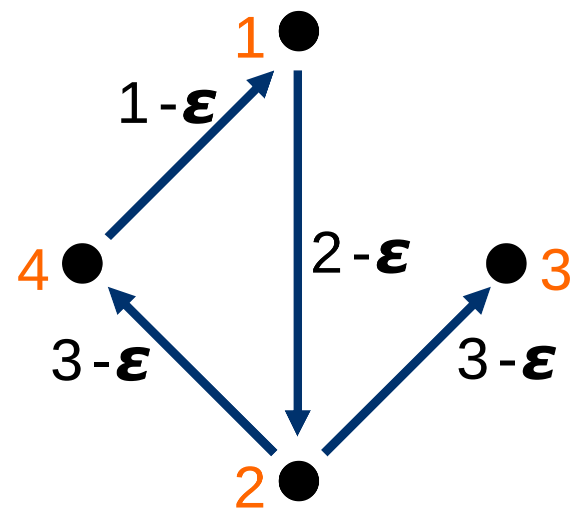
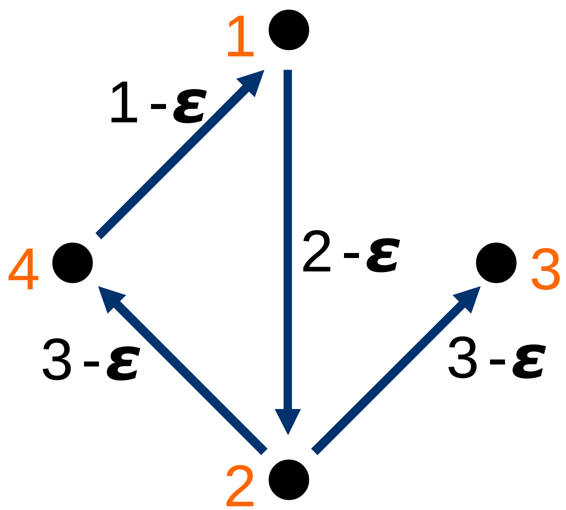
Remark: profits are homogeneous and
costs are ε close to homogeneous



Example without Nash Network

The cost / payoff structure



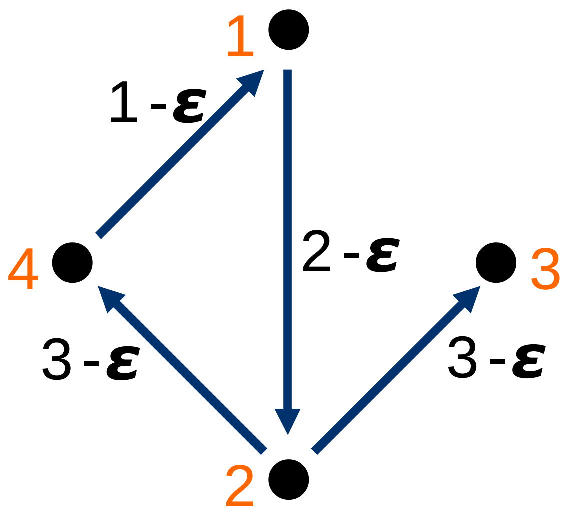
other links to 1 cost 1 + ε
other links to 2 cost 2 + ε
other links to 3, 4 cost 3 + ε
profits v_{ij} = 1 for all i and j

The arguments (part A)

In any Nash network
agent 3 and agent 4
would either play {2} or Φ.

Example without Nash Network

The cost / payoff structure



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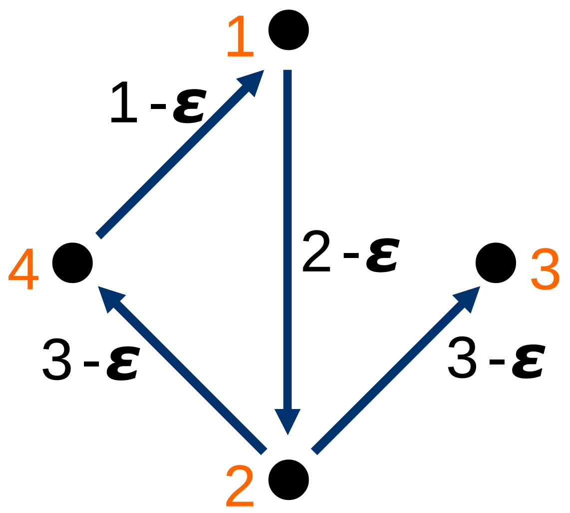
profits v_{ij} = 1 for all i and j

The arguments (part A)

In any Nash network
agent 3 and agent 4
would either play {2} or Φ.
If agent 4 plays {2},
then agent 1 plays {4}.

Example without Nash Network

The cost / payoff structure



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other links to 3, 4 cost 3 + ε

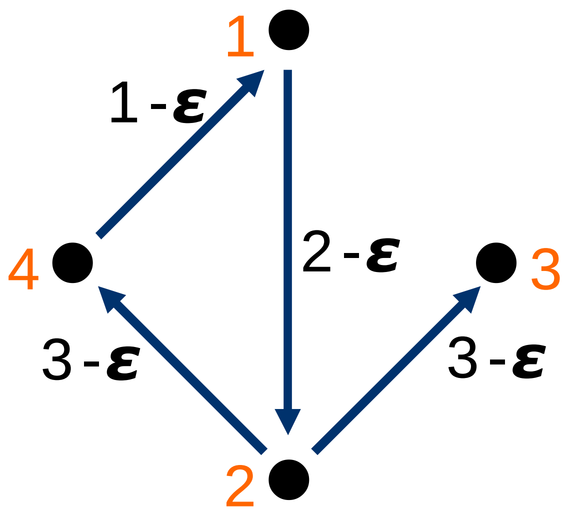
profits v_{ij} = 1 for all i and j

The arguments (part A)

In any Nash network
agent 3 and agent 4
would either play {2} or Φ.
If agent 4 plays {2},
then agent 1 plays {4}.
Then agent 2 plays {1},
because agent 3 never plays {1}.

Example without Nash Network

The cost / payoff structure



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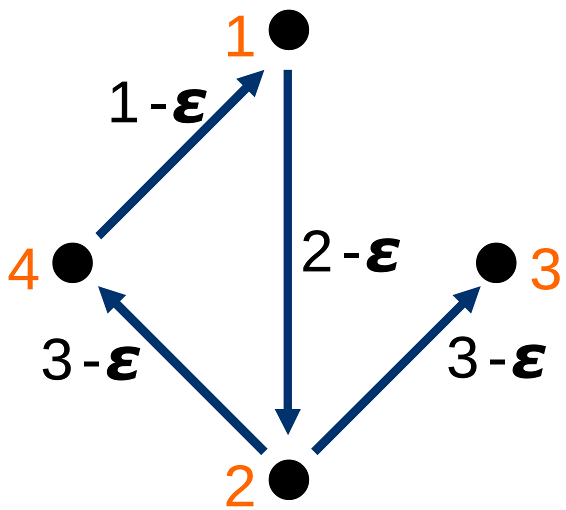
profits v_{ij} = 1 for all i and j

The arguments (part A)

In any Nash network
agent 3 and agent 4
would either play {2} or Φ.
If agent 4 plays {2},
then agent 1 plays {4}.
Then agent 2 plays {1},
because agent 3 never plays {1}.
Then agent 3 plays {2}.

Example without Nash Network

The cost / payoff structure



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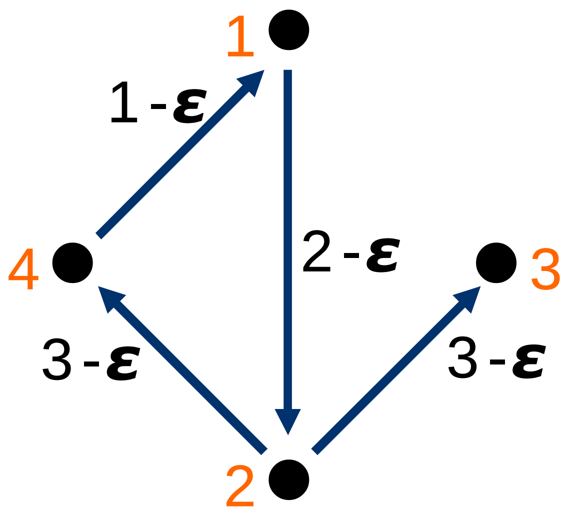
other links to 3, 4 cost 3+ε
profits v_{ij} = 1 for all i and j

The arguments (part A)

In any Nash network
agent 3 and agent 4
would either play {2} or Φ.
If agent 4 plays {2},
then agent 1 plays {4}.
Then agent 2 plays {1},
because agent 3 never plays {1}.
Then agent 3 plays {2}.
Then agent 4 should play Φ.

Example without Nash Network

The cost / payoff structure



other links to 1 cost $1+\varepsilon$

other links to 2 cost $2+\varepsilon$

other links to 3, 4 cost $3+\varepsilon$

profits v_{ij} = 1 for all i and j

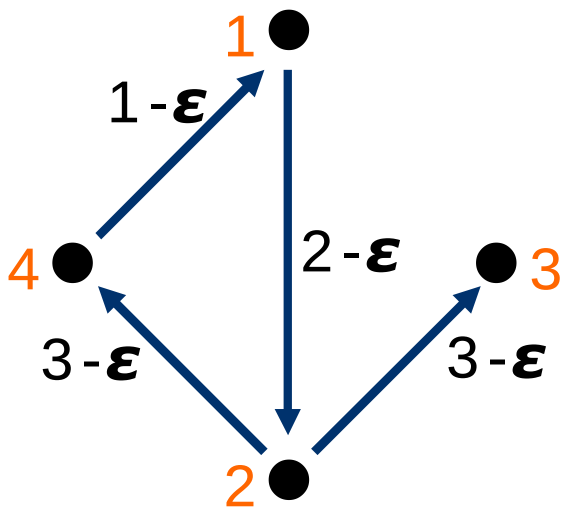
The arguments (part A)

In any Nash network
agent 3 and agent 4
would either play {2} or Φ.
If agent 4 plays {2},
then agent 1 plays {4}.
Then agent 2 plays {1},
because agent 3 never plays {1}.
Then agent 3 plays {2}.
Then agent 4 should play Φ.

A contradiction

Example without Nash Network

The cost / payoff structure



other links to 1 cost 1+ε

other links to 2 cost 2+ε

other links to 3, 4 cost 3+ε

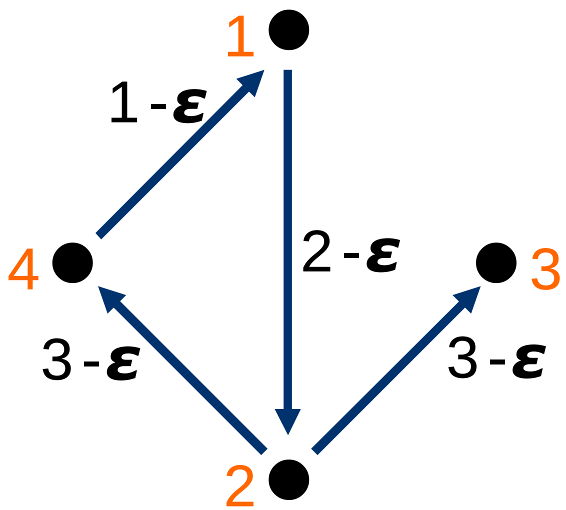
profits v_{ij} = 1 for all i and j

The arguments (part B)

In any Nash network
agent 3 and agent 4
would either play {2} or Φ.
If agent 4 plays Φ,
then agent 1 plays S containing 4.

Example without Nash Network

The cost / payoff structure



other links to 1 cost 1+ε

other links to 2 cost 2+ε

other links to 3, 4 cost 3+ε

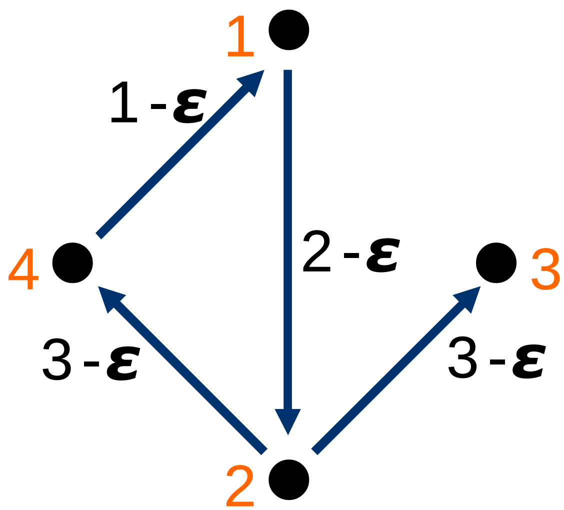
profits v_{ij} = 1 for all i and j

The arguments (part B)

In any Nash network
agent 3 and agent 4
would either play {2} or **Φ**.
If agent 4 plays **Φ**,
then agent 1 plays *S* containing 4.
Then agent 2 plays {1}.

Example without Nash Network

The cost / payoff structure



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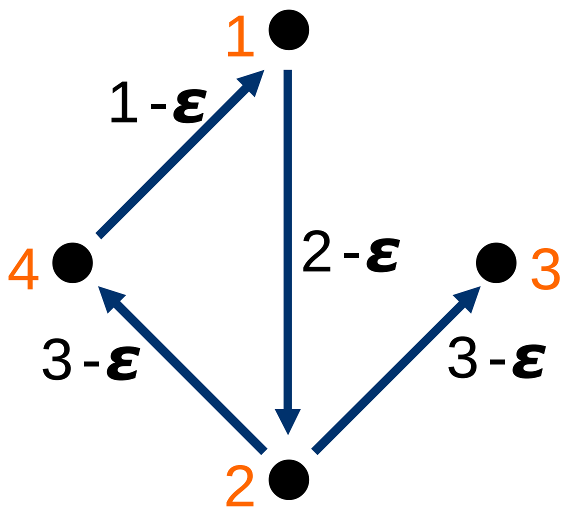
profits v_{ij} = 1 for all i and j

The arguments (part B)

In any Nash network
agent 3 and agent 4
would either play {2} or Φ.
If agent 4 plays Φ,
then agent 1 plays S containing 4.
Then agent 2 plays {1}.
Then agent 3 plays {2}.

Example without Nash Network

The cost / payoff structure



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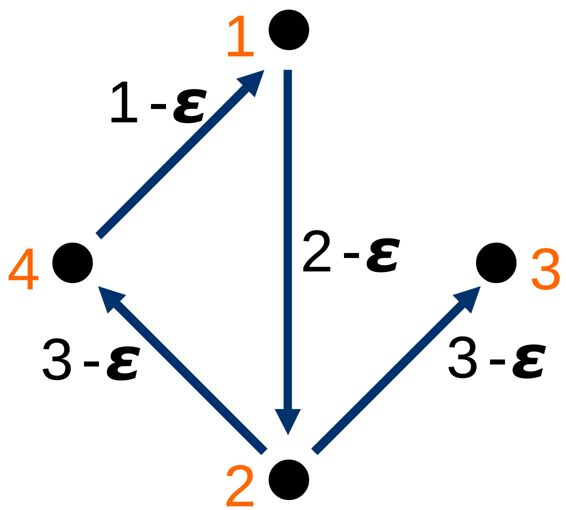
profits v_{ij} = 1 for all i and j

The arguments (part B)

In any Nash network
agent 3 and agent 4
would either play {2} or Φ.
If agent 4 plays Φ,
then agent 1 plays S containing 4.
Then agent 2 plays {1}.
Then agent 3 plays {2}.
Then agent 1 plays {3,4}.

Example without Nash Network

The cost / payoff structure



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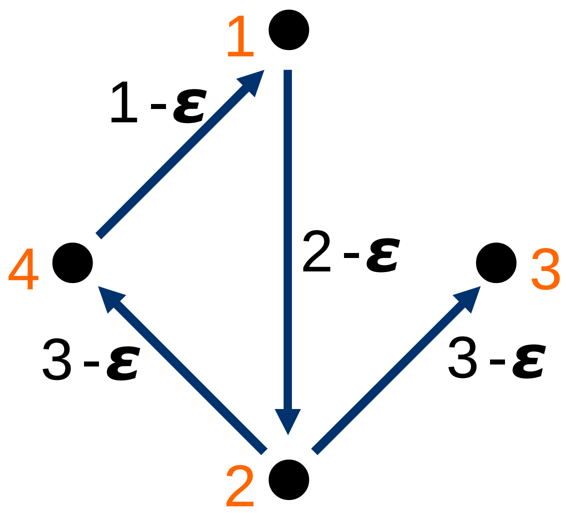
profits v_{ij} = 1 for all i and j

The arguments (part B)

In any Nash network
agent 3 and agent 4
would either play {2} or *Φ*.
If agent 4 plays *Φ*,
then agent 1 plays *S* containing 4.
Then agent 2 plays {1}.
Then agent 3 plays {2}.
Then agent 1 plays {3,4}.
Then agent 4 should play {2}.

Example without Nash Network

The cost / payoff structure



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In any Nash network
agent 3 and agent 4
would either play {2} or Φ.
If agent 4 plays Φ,
then agent 1 plays S containing 4.
Then agent 2 plays {1}.
Then agent 3 plays {2}.
Then agent 1 plays {3,4}.
Then agent 4 should play {2}.

Again a contradiction

Concluding Remarks

Independently, an alternative proof for our theorem is given by:

- P. Billand, C. Bravard, S. Sarangi (2008): Existence of Nash Networks in one-way flow models. *Economic Theory* 37: 491–507.

Yet another proof, based directly on Billand et al., is given in:

- J. Derks, M. Tennekes (2008): A note on the existence of Nash networks in one-way flow models. *Economic Theory* 41:515–522.

The results presented can be found in:

- J. Derks, J. Kuipers, M. Tennekes, F. Thuijsman (2009): Existence of Nash networks in the one-way flow model of network formation. In: S.K. Neogy, A.K. Das, R.B. Bapat (eds.) *Modeling, Computation and Optimization*, World Scientific, pp 9-20.
- M. Tennekes (2010): *Network Formation Games.* PhD Thesis, Maastricht University.

Thank you for your attention!

The paper and presentation will be available at my homepage.

Comments are welcome any time.