One-Way Flow Nash Networks

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Outline

- The Model of One-Way Flow Networks
- An Existence Result and a Structural Observation
- A Dynamic Procedure of Local Actions
- A Counterexample
The Model of One-Way Flow Networks

Network Formation Game \((N,v,c)\)

- \(N = \{1,2,3,...,n\}\)
- \(v_{ij} \geq 0\) is the profit for agent \(i\) for being connected to agent \(j\)
- \(c_{ij} \geq 0\) is the cost for agent \(i\) for being \textit{directly} connected to agent \(j\)
The Model of One-Way Flow Networks

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Example of a one-way flow network \(g\)
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Example of a one-way flow network \(g\)

Agent 1 is connected to agents 3, 4, 5 and 6 and obtains profits \(v_{13}, v_{13}, v_{14}, v_{15}, v_{16}\). Agent 1 is \textit{not} connected to agent 2. Agent 1 has to pay \(c_{13}\) for the link (3,1).
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The payoff \(\pi_1(g)\) for agent 1 is

\[ \pi_1(g) = v_{13} + v_{14} + v_{15} + v_{16} - c_{13}. \]
The Model of One-Way Flow Networks

Network Formation Game $(N, \nu, c)$

- $N = \{1, 2, 3, \ldots, n\}$
- $\nu_{ij} \geq 0$ is the profit for agent $i$ for being connected to agent $j$
- $c_{ij} \geq 0$ is the cost for agent $i$ for being *directly* connected to agent $j$

More generally:

$$\pi_i(g) = \sum_{j \in N_i(g)} \nu_{ij} - \sum_{j \in Nd_i(g)} c_{ij}$$

where $N_i(g)$ is the set of agents that $i$ is connected to in $g$, and where $Nd_i(g)$ is the set of agents that $i$ is *directly* connected to in $g$. 
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where \(N_i(g)\) is the set of agents that \(i\) is connected to in \(g\), and where \(Nd_i(g)\) is the set of agents that \(i\) is directly connected to in \(g\).

Our model is mainly based on:
The Model of One-Way Flow Networks

An action for agent $i$ is any subset $S$ of $N\backslash\{i\}$ indicating the set of agents that $i$ connects to directly.
The Model of One-Way Flow Networks

An **action** for agent $i$ is any subset $S$ of $\mathbb{N}\setminus\{i\}$ indicating the set of agents that $i$ connects to directly.

A network $g$ is a **Nash network** if each agent $i$ is playing a best response in terms of his individual payoff $\pi_i(g)$. 
“A Beautiful Mind”

John F. Nash

A Closer Look at Nash Networks

A network $g$ is a Nash network if for each agent $i$

$$\pi_i(g) \geq \pi_i(g_{-i} + \{(j,i) : j \in S\})$$

for all subsets $S$ of $N\{i\}$.

Here $g_{-i}$ denotes the network derived from $g$ by removing all direct links of agent $i$. 
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A set $S$ that maximizes the right-hand side of above expression is called a best response for agent $i$ to the network $g$.

In a Nash network each agent is linked to a best response.
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In a Nash network each agent is linked to a best response.

Remark: If $c_{ik} > \Sigma_{j \neq i} v_{ij}$ for all agents $k \neq i$, then the only best response for agent $i$ is the empty set $\emptyset$. 
Owner-Homogeneous Costs

For each agent $i$ all links are equally expensive: $c_{ij} = c_i$ for all $j$. 
Owner-Homogeneous Costs

For each agent \( i \) all links are equally expensive: \( c_{ij} = c_i \) for all \( j \).

Observation for owner-homogeneous costs
If link \((j,k)\) exists in \( g \),
then for agent \( i \neq j,k \), linking with \( k \)
is at least as good as linking with \( j \).

“Downstream Efficiency”
Lemma

For any network formation game \((N, \nu, c)\) with owner-homogeneous costs and with \(c_i \leq \sum_{j \neq i} v_{ij}\) for all agents \(i\), all cycle networks are Nash networks.
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Proof by examining agent 1:
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Proof by examining agent 1:

When removing \((2,1)\) agent 1 looses profits from agents 2, 3, 4, 5, 6.
Lemma

For any network formation game \((N, v, c)\) with owner-homogeneous costs and with \(c_i \leq \sum_{j \neq i} v_{ij}\) for all agents \(i\), all cycle networks are Nash networks.

Proof by examining agent 1:

When replacing \((2,1)\) by \((4,1)\) agent 1 looses profits from agents 2 and 3.
Lemma

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Proof by examining agent 1:

When adding \((4,1)\) agent 1 pays an additional cost of \(c_{14}\).
Lemma

For any network formation game \((N,v,c)\) with owner-homogeneous costs and with \(c_i \leq \sum_{j \neq i} v_{ij}\) for all agents \(i\), all cycle networks are Nash networks.

Proof by examining agent 1:

Hence \(\{2\}\) is a best response for agent 1.
Theorem

For any network formation game \((N,v,c)\) with owner-homogeneous costs, a Nash network exists.
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**Proof** by induction to the number of agents \(n\):

If \(n = 1\), then the trivial network is a Nash network.
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Proof by induction to the number of agents \(n\):

If \(n = 1\), then the trivial network is a Nash network.

Induction hypothesis: Nash networks exist for all network games with less than \(n\) agents.
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For any network formation game \((N, v, c)\) with owner-homogeneous costs, a Nash network exists.

**Proof** by induction to the number of agents \(n\):

If \(n = 1\), then the trivial network is a Nash network.

Induction hypothesis: Nash networks exist for all network games with less than \(n\) agents.

Suppose that \((N, v, c)\) is a network game with \(n\) agents for which no Nash network exists.
Recall the Lemma:

For any network formation game \((N,v,c)\) with owner-homogeneous costs and with \(c_i \leq \sum_{j \neq i} v_{ij}\) for all agents \(i\), all cycle networks are Nash networks.
Proof Continued:

Hence there is at least one agent $i$ with $c_i > \Sigma_{j\neq i} v_{ij}$.
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Hence there is at least one agent $i$ with $c_i > \sum_{j \neq i} v_{ij}$. W.l.o.g. this agent is agent $n$. 
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Hence there is at least one agent $i$ with $c_i > \sum_{j \neq i} v_{ij}$. W.l.o.g. this agent is agent $n$.

Consider $(N', v', c')$ with $N' = N \setminus \{n\}$ and with $v$ and $c$ restricted to agents in $N'$. 
Proof Continued:

Hence there is at least one agent $i$ with $c_i > \sum_{j \neq i} v_{ij}$.
W.l.o.g. this agent is agent $n$.
Consider $(N', v', c')$ with $N' = N \setminus \{n\}$
and with $v$ and $c$ restricted to agents in $N'$.
Let $g'$ be a Nash network in $(N', v', c')$ (induction hypothesis).
Proof Continued:

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Then by assumption $g'$ is no Nash network in $(N, v, c)$. 
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Let $g'$ be a Nash network in $(N', v', c')$ (induction hypothesis).
Then by assumption $g'$ is no Nash network in $(N, v, c)$.
Therefore there is an agent $i$ for whom the links in $g'$ are no best response in $(N, v, c)$.
Proof Continued:

Hence there is at least one agent $i$ with $c_i > \sum_{j \neq i} v_{ij}$.

W.l.o.g. this agent is agent $n$.

Consider $(N', v', c')$ with $N' = N \setminus \{n\}$
and with $v$ and $c$ restricted to agents in $N'$.

Let $g'$ be a Nash network in $(N', v', c')$ (induction hypothesis).
Then by assumption $g'$ is no Nash network in $(N, v, c)$.

Therefore there is an agent $i$
for whom the links in $g'$ are no best response in $(N, v, c)$.
This agent $i$ can not be agent $n$;
so w.l.o.g. this agent is agent 1
and he has a best response $T$ with $n \in T$
and therefore $c_1 \leq v_{1n}$,
because agent $n$ is not linked to anyone else.
Proof Continued:

Now recall that, by downstream efficiency, for any other agent $i$ linking to agent 1 would be at least as good as linking to agent $n$.

Define $v_{ij}^* = \begin{cases} 
    v_{ij} & \text{for } j \neq 1 \\
    v_{i1} + v_{in} & \text{for } i \neq 1, j = 1 \\
    v_{11} + v_{1n} - c_1 & \text{for } i = 1, j = 1 
\end{cases}$
Proof Continued:

Now recall that, by downstream efficiency, for any other agent $i$ linking to agent $1$ would be at least as good as linking to agent $n$.

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\end{cases}$

Now $\pi^*_i(g) = \pi_i(g + (n,1))$ for any network $g$ on $N'$ and for any agent $i$ in $N'$. 

Proof Continued:

Now recall that, by downstream efficiency, for any other agent $i$ linking to agent 1 would be at least as good as linking to agent $n$.

Define $v_{ij}^* = \begin{cases} v_{ij} & \text{for } j \neq 1 \\ v_{i1} + v_{in} & \text{for } i \neq 1, j = 1 \\ v_{11} + v_{1n} - c_1 & \text{for } i = 1, j = 1 \end{cases}$

Now $\pi_i^*(g) = \pi_i(g + (n,1))$ for any network $g$ on $N'$ and for any agent $i$ in $N'$.

By the induction hypothesis the game $(N',v^*,c')$ has a Nash network $g^*$.
**Proof Continued:**

In \( g^* \) all agents in \( N' \) play best responses w.r.t. \((N',v^*,c')\) because \( g^* \) is a Nash network.

By the way that \( v^* \) was defined, this implies that w.r.t. \((N,v,c)\) in \( g^* \) all agents in \( N' \) play best responses. And we still have that w.r.t. \((N,v,c)\) the only best response for agent \( n \) is to play \( \Phi \) in any network, particularly in \( g^* \).

Hence \( g^* \) is a Nash network in \((N,v,c)\), This contradicts the initial assumption that there is no Nash network in \((N,v,c)\). ■
Observation

For each network formation game with owner-homogeneous costs there exists at least one Nash network with at most one cycle and with every vertex having an out-degree of at most 1...

... but there may well be other Nash networks as well.
Example

This network $g_1$ is a Nash network where

$$\pi_i(g_1) = 4, \quad \pi_j(g_1) = 5, \quad \pi_k(g_1) = 5.$$  

Notice that agent $i$ and agent $k$ have only one best response, but agent $j$ is indifferent between linking to $i$ or linking to $k$. 
Example

If agent $j$ replaces the link to $i$ by one to $k$, then we get the network $g_2$ where the payoffs are still the same $\pi_i(g_2) = 4$, $\pi_j(g_2) = 5$, $\pi_k(g_2) = 5$.

However, $g_2$ is no Nash network since agent $i$ can improve his payoff by removing the link to $k$. 
Example

If agent \( i \) removes the link to \( k \), then we get the cycle network \( g_3 \) which is a Nash network with

\[
\pi_i(g_3) = 5, \quad \pi_j(g_3) = 5, \quad \pi_k(g_3) = 5.
\]
A Dynamic Procedure of Local Actions

Recall that an action for agent $i$ is any subset $S$ of $N \setminus \{i\}$ indicating the set of agents that $i$ connects to directly.

A local action for agent $i$ in a dynamic context is one of these:
- not changing anything in the network
- deleting one link $(j,i)$
- adding one link $(k,i)$
- replacing one link $(j,i)$ by another link $(k,i)$

A network $g$ is a local Nash network if each agent $i$ is playing a best local response in terms of his individual payoff $\pi_i(g)$. 
A Dynamic Procedure of Local Actions

Let $g_t$ be the network at stage $t$ and suppose that agent $i$ plays a local action $a$ that leads to the network $g_{t+1}$ then we define action $a$ to be a good local response if

$$\pi_i(g_{t+1}) \geq \pi_i(g_t)$$
A Dynamic Procedure of Local Actions

Start with an arbitrary network $g_1$ at stage $1$. Let $g_t$ be the network at stage $t$. If $g_t$ is a local Nash network with maximum outdegree 1, then stop. Otherwise, choose an agent $i$ at random and let $i$ play a random good local response, which leads to the network $g_{t+1}$ to be examined at stage $t+1$.

Theorem
This procedure ends in a global Nash network with probability 1.

(Proof skipped here)
Example without Nash Network

For network formation games \((N,v,c)\) with *heterogeneous* costs, Nash networks do not need to exist.
Example without Nash Network

For network formation games \((N, v, c)\) with heterogeneous costs, Nash networks do not need to exist.

A heterogeneous costs structure

other links to agent 1 cost \(1+\epsilon\)
other links to agent 2 cost \(2+\epsilon\)
other links to agents 3 and 4 cost \(3+\epsilon\)

profits \(v_{ij} = 1\) for all \(i\) and \(j\)
Example without Nash Network

For network formation games \((N,v,c)\) with heterogeneous costs, Nash networks do not need to exist.

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profits \(v_{ij} = 1\) for all \(i\) and \(j\)

Remark: profits are homogeneous and costs are \(\varepsilon\) close to homogeneous
Example without Nash Network

The cost/payoff structure

The arguments (part A)

In any Nash network, agent 3 and agent 4 would either play \{2\} or \emptyset.

other links to 1 cost 1+\epsilon
other links to 2 cost 2+\epsilon
other links to 3, 4 cost 3+\epsilon
profits \( v_{ij} = 1 \) for all \( i \) and \( j \)
Example without Nash Network

The cost/payoff structure

The arguments (part A)

In any Nash network
agent 3 and agent 4
would either play \{2\} or \Phi.
If agent 4 plays \{2\},
then agent 1 plays \{4\}.

other links to 1 cost 1+\epsilon
other links to 2 cost 2+\epsilon
other links to 3, 4 cost 3+\epsilon
profits \( v_{ij} = 1 \) for all \( i \) and \( j \)
Example without Nash Network

The cost/payoff structure

- Other links to 1 cost 1+ε
- Other links to 2 cost 2+ε
- Other links to 3, 4 cost 3+ε
- Profits $v_{ij} = 1$ for all $i$ and $j$

The arguments (part A)

In any Nash network, agent 3 and agent 4 would either play $\{2\}$ or $\emptyset$.
If agent 4 plays $\{2\}$, then agent 1 plays $\{4\}$.
Then agent 2 plays $\{1\}$, because agent 3 never plays $\{1\}$. 
Example without Nash Network

The cost/payoff structure

In any Nash network agent 3 and agent 4 would either play \{2\} or \emptyset.
If agent 4 plays \{2\}, then agent 1 plays \{4\}.
Then agent 2 plays \{1\}, because agent 3 never plays \{1\}.
Then agent 3 plays \{2\}.

profits $v_{ij} = 1$ for all $i$ and $j$
Example without Nash Network

The cost/payoff structure

The arguments (part A)

In any Nash network agent 3 and agent 4 would either play \{2\} or \Phi.

If agent 4 plays \{2\}, then agent 1 plays \{4\}.

Then agent 2 plays \{1\}, because agent 3 never plays \{1\}.

Then agent 3 plays \{2\}.

Then agent 4 should play \Phi.

other links to 1 cost 1+\varepsilon
other links to 2 cost 2+\varepsilon
other links to 3, 4 cost 3+\varepsilon
profits \(v_{ij} = 1\) for all \(i\) and \(j\)
Example without Nash Network

The cost/payoff structure

The arguments (part A)

In any Nash network
agent 3 and agent 4
would either play \{2\} or \emptyset.
If agent 4 plays \{2\},
then agent 1 plays \{4\}.
Then agent 2 plays \{1\},
because agent 3 never plays \{1\}.
Then agent 3 plays \{2\}.
Then agent 4 should play \emptyset.

A contradiction
Example without Nash Network

The cost/payoff structure

In any Nash network agent 3 and agent 4 would either play \{2\} or Φ. If agent 4 plays Φ, then agent 1 plays S containing 4.
Example without Nash Network

### The cost/payoff structure

The cost/payoff structure is visualized in the diagram. The nodes represent agents, and the arrows indicate the costs associated with each link.

- Node 1 and Node 3 have a Cost of 1-\(\varepsilon\)
- Node 4 has a Cost of 2-\(\varepsilon\)
- Node 2 and Node 4 have a Cost of 3-\(\varepsilon\)

### The arguments (part B)

In any Nash network, agent 3 and agent 4 would either play \{2\} or \Phi. If agent 4 plays \Phi, then agent 1 plays \(S\) containing 4. Then agent 2 plays \{1\}.

Other links:
- Other links to Node 1 cost \(1+\varepsilon\)
- Other links to Node 2 cost \(2+\varepsilon\)
- Other links to Nodes 3 and 4 cost \(3+\varepsilon\)

Profits:
- \(v_{ij} = 1\) for all \(i\) and \(j\)
Example without Nash Network

The cost/payoff structure

1

1-ε

4

2-ε

3

3-ε

2

The arguments (part B)

In any Nash network
agent 3 and agent 4
would either play \{2\} or Φ.
If agent 4 plays Φ,
then agent 1 plays S containing 4.
Then agent 2 plays \{1\}.
Then agent 3 plays \{2\}.

other links to 1 cost 1+ε
other links to 2 cost 2+ε
other links to 3, 4 cost 3+ε
profits v_{ij} = 1 for all i and j
Example without Nash Network

The cost/payoff structure

- Other links to 1 cost $1 + \varepsilon$
- Other links to 2 cost $2 + \varepsilon$
- Other links to 3, 4 cost $3 + \varepsilon$
- Profits $v_{ij} = 1$ for all $i$ and $j$

The arguments (part B)

In any Nash network, agent 3 and agent 4 would either play $\{2\}$ or $\Phi$.
- If agent 4 plays $\Phi$, then agent 1 plays $S$ containing 4.
- Then agent 2 plays $\{1\}$.
- Then agent 3 plays $\{2\}$.
- Then agent 1 plays $\{3, 4\}$. 
Example without Nash Network

The cost/payoff structure

The arguments (part B)

In any Nash network
agent 3 and agent 4
would either play \{2\} or \emptyset.
If agent 4 plays \emptyset,
then agent 1 plays \mathcal{S} containing 4.
Then agent 2 plays \{1\}.
Then agent 3 plays \{2\}.
Then agent 1 plays \{3, 4\}.
Then agent 4 should play \{2\}. 
Example without Nash Network

The cost/payoff structure

The arguments (part B)

In any Nash network
agent 3 and agent 4 would either play \{2\} or $\emptyset$.
If agent 4 plays $\emptyset$, then agent 1 plays $S$ containing 4.
Then agent 2 plays \{1\}.
Then agent 3 plays \{2\}.
Then agent 1 plays \{3,4\}.
Then agent 4 should play \{2\}.

Again a contradiction
Concluding Remarks

Independently, an alternative proof for our theorem is given by:


Yet another proof, based directly on Billand et al., is given in:


The results presented can be found in:

Thank you for your attention!

The paper and presentation will be available at my homepage.

Comments are welcome any time.