Master Thesis

A Cooperative Game Theory Application in The Blackbird Broods Food Allocation

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Abstract

We study food allocation in bird broods from the perspective of cooperative game theory. We want to explore whether or not food distribution data fit into the known solution concepts of cooperative game theory. A first issue to be handled is the fact that in the bird brood data we only see the solutions, while the starting position, the game, is not immediately clear. As such we need to reconstruct the game from the solutions given. A second issue is that there are many different solution concepts (e.g. Shapley value, nucleolus, etc) and we want to analyze which of these fits best. Most interesting is to specifically address the properties that lead to these solutions, because these would be most useful in finding a motivation for the specific solution concept found in nature.

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I dedicate this thesis to my family: especially to my father Noftiman Nasir, to my mother Metty Rosita, to my sisters Rara, Mala, and Dinda, and to my husband Zuhairi Sanofi. Their patience and support during the process are my motivation to continue and finalize my thesis.

Chapter 1

Introduction

Presently there seems to be hardly any applications of cooperative game theory in the field of biology. We want to survey what applications of cooperative game theory in biology are presently known in the contemporary literature, and whether one of these has a connection to the problem of food allocation, because of the availability of a large dataset on blackbirds food distributions gathered by Professor Scott Forbes and his team from Winnipeg University. The first subsection will give a brief description about the problem of bird brood allocations, while the second subsection will provide the state of the art of applications of cooperative game theory within the field of biology.

1.1 Bird Broods Fair Allocation

Here we decribe the nature behind the bird broods fair allocation problem and explain why we would like to relate it to the Cooperative Game Theory concept.

1.1.1 Birds vs Human: The Nature of Family

In the real world, where there will never be any certainty nor sufficient informations about the future, parents hedge their bets in building unpredictable family life. Parents often do not know in advance what resources will be available, and thus are investing in offspring in a climate of uncertainty. Setting the initial family size too large risks future food shortfalls; a family size too small may result in costs of lost opportunities if food proves plentiful. [16] Scott Forbes in his book titled *A Natural History of Families* [1] mentions that there are some similarities (as well as differences) of birds and human especially in parenting. Both cases have a problem of what is called unholy parenting, where parents are programmed to provide less, while children often ask for more. In the end, they are both left unsatisfied. However, study has shown that this is an equilibrium in nature, that it is, indeed, just how it should be.

Families are bound together and share common genetic interest, thus are supposed to lower the barrier of cooperation and minimize the conflict, making it an ideal model for the evolution of cooperation. However, conflict will still exist and usually interest people the most. Forbes in his paper *Sibling Symbiosis in Nestling Birds* [2] gave the fact that a November 2006 search of the Web of Science revealed 334 citations for the keywords "nestling conflict" or "nestling competition", and only 11 for "nestling cooperation" or "nestling mutualism". It is either a parent-offspring conflict or a conflict between the offsprings, in terms of food, shelter, space, warmth, etc.

Asymmetric sibling rivalry (Forbes and Glassey 2000), where stronger, older sibs outcompete younger broodmates, is common in birds and mammals and usually stems from parental (usually maternal) manipulations of offspring morphology and behavior (Mock and Parker 1997; Hudson and Trillmich 2008). It is generated by parents imposing handicaps upon some of their offspring while conferring advantages to others: effectively, parents play favourites. For example, mothers may make some eggs or newborns bigger than others (Slagsvold et al. 1984; Rdel et al. 2008; Forbes and Wiebe 2010); or they may fortify some progeny with extra hormones making them more successful in begging competitions (Groothuis et al. 2005; Sockman et al. 2006) or add immune system components and/or antioxidants, conferring resistance to pathogens or cellular damage (Royle et al. 2001; Saino et al. 2002). But most importantly, mothers often create age differences among contemporary progeny via birth or hatching asynchrony (Lack 1947; Magrath 1990; Trillmich and Wolf 2008). All of these parental manipulations serve to render some offspring more equal than others. [16]

In birds, hatching asynchrony (Glassey and Forbes 2002) creates a brood hierarchy dividing the progeny into castes of advantaged core and disadvantaged marginal offspring (Mock and Forbes 1995; Forbes et al. 1997; Forbes 2011). Most often the core brood consists of two or more nestlings that are the same size and age, the rough equivalent of multiple births in humans, and one or more marginal offspring hatch one or more days after the core brood. Core progeny enjoy an advantage over their marginal counterparts in sibling competitions for limited parental resources such as food, and generally exhibit higher growth and survival over the period of parental care. When food is short, marginal progeny are the first to perish, becoming victims of socially enforced starvation and/or sibling aggression (review in Mock and Parker 1997). This division into core and marginal elements structures the avian family, and parents now make two choices at the outset of breeding: what size of family to have, and how is this family structured.

How about in human? In a movie inspired by the book of Jodi Picoult titled **My Sister Keeper**, a girl named Anna plays a role as a saviour sibling; she was born in order to dispute blood from her umbilical cord, or any other bodily substance needed (such as kidney), as a part of the treatment to save her sibling Kate from death by cancer. The story points out parents' dilemma of whether it is morally correct to do whatever it takes to save a child's life, even if it means infringing upon the rights of another child. The story succesfully shows what it means to be a good parent, good sister, and a good person in general. But how to determine good? Just like how we determine what is fair; fair for whom and fair from whose point of view often give different results.

1.1.2 Why study Blackbirds?

The same thing also happens in blackbirds: not all offsprings are created equal, in other words, there is often a competitive asymmetry between the offsprings. Chick that is hatched last, often is the one to die first. It could be because parents, just like the story of Kate and Anna above, created core and marginal offspring intentionally before giving birth; making the children have different roles in life. This phenomenon is called parental favoritism: parents choose which children they hatch first, thus having bigger probability to survive compared to the siblings which will be hatched a day after. In the case of blackbirds which have in average one to five children during hatching, it is usually the case that parents cannot nurture all the children and they know that one or some of them will die young. Then why would the parents still give birth to them if parents already know the child will not survive after some period of time? That is another phenomenon called parental optimitism; a strategy where parents set an initially optimistic family size and trim downward (brood reduction) as unfolding food conditions warrant.

Birds are long choosen to be the model system to study parental favoritism and parental optimitism. The dynamics are easiest to be observed in birds in general, as they do not hide their progeny in unaccessible wombs (as in human) or out of sight (underwater, as generally seen in fish) [1]. Blackbirds, specifically, compromise in many respected model system. They are easily accessible and occur in very large number. They also nest close to the ground, enabling researchers to directly check what happens. Previous research shows that in blackbirds, we can really see the difference between core and marginal offspring clearly as they do not seem to be disturbed by the camera put near their broods. Thus blackbirds are chosen to be object of this food allocation study.

Quoting Forbes, marginal nestlings of red-wing blackbirds lag behind their larger core siblings in both size and development through no fault of their own, and the manipulative parents may even compound their woes by giving an extra dose of testosterone as a privilege to the core brood, making them more belligerent at feeding time. Previous study shows an optimal intermediate solution for blackbirds, where the parents objective is to avoid broods being too small. Thus parents intentionally do two things to narrow the competitive edge between core and marginal chicks by giving steroids preferentially to the marginal one, or making the last hacthlings larger.

As Forbes also illustrates in his book, yellow-headed blackbirds provide a tidy illustration of parental optimism at work. They breed in prairie wetlands accross central North America. Male chicks are bigger than females, yielding a problem when food is short as big bodies are more expensive to maintain. Thus parents often put one male in the core brood while any additional males are placed in the marginal brood and can be easily eliminated when food resource is scarce.

As a result of living in such an unpredictable world, parents already realized that they need a guarantee so that their broods will not be too small in risk of dying when situation gets worse (i.e. food resource is scarce or weather goes bad). By focusing on the core offspring while still having marginal one as a back-up plan, they could inrease the probability of success. Imagine if during a bad season where food is very limited, parents only have one core chick and it still dies even if they take really good care and give it all the food they have. By having at least one core and one marginal offspring, if something happens to the first one (fail to hatch or perish early), parents have an 'insurance': they could raise the second one. If something happens to the second during the bad situation, as long as the first one stays healthy, they give little to no care as the second is marginal. Doing so, the marginal one already serves its role as a facilitation to the core, by being a layer where the core could huddle, preventing them from heat or cold. The probability of both chicks being failed is of course smaller. These illustrations are known as the brood reduction policy in birds.

Being a smart parent, having at least one core and one marginal in a good season where food resource are abundant is also best for them as they could get a 'bonus' by raising the marginal offspring, taking into account that even the marginal offspring gets less food than the core one, this food is already enough for the chick to stay alive. This core-marginal solution indeed gives parents the most incentives of all. Therefore bird parents routinely start with more progeny than will ever survive to independence, as eggs and embryos are cheap while subsequent parental care is much more costly.

1.1.3 Why Cooperative Game?

"Life histories are shaped by trade-offs. One key trade-off is the principle of allocation. Resources are finite and compromises necessary." -Scott Forbes, *A Natural History of Families*, 2005 [1]

Evolutionary biologists have a rich tradition of borrowing analytical tools from economists to address a diverse array of problems in nature. Maximizing the number of surviving offsprings (instead of maximizing offspring numbers) are usually the object of evolutionary game theory, as these children will continue the family legacy. Larger broods are disfavored if such strategy leaves with malnourished infant with poor prospects of survival. The results are fixed pie: there is only a fixed amount to be shared and in order for one person to win, the other must lose, while the pie size cannot be expanded. Problem is, how to divide this pie among the current broods, cooperatively and fairly?

In birds, where often food is the whole story, unequal allocation of resources is a consequence of parents playing favorites. This does not mean that it is not fair to the marginal chick getting less than the core one. Having, let's say, four chicks in one brood with two cores (one male, one female) and two marginals (one male, one female) during a bad situation where food is limited, we could observe what is the fair allocation of dividing a fixed amount of food to four different chicks (each with different role and characteristic). Moreover, we could also see whether in fact each chick could get what it is supposed to get based on cooperative game solution concept, to be able to survive. If for example one marginal male chick perishes in the end, we could check whether it is because of the chick cannot get sufficient amount of food than its fair shaire according to a specific solution concept.

The available bird broods data that are collected during earlier research conducted by Scott Forbes in Canada, could be used to compare food sharing in bird broods using some known solution concept in cooperative games. More about this brood data will be described in the next chapter.

1.2 Previous Studies

A study that has strong relation with our problem is described in Scott Forbes book titled A Natural History of Families [1], as mentioned earlier in our problem description; especially on its second and third chapter. Therefore this thesis will use similar terms and definitions as mentioned in the Forbes book.

1.2.1 Study on Bird Broods Fair Allocation Problem

Recent works by Forbes (2009) is using a financial tool to the study of parental investment in birds, as normally the most important investment any organism makes is in its offspring. A key dimension of any investment decision including how much to invest in offspring is how to balance risk and reward for which portfolio theory offers a broad set of analytical tools [6]. A primary difficulty for the biologist lies in how to translate the economic models to a biological setting. The tool he used is called financial beta and is well-known to the study of parental investment, derived from the capital asset pricing model of modern portfolio theory. Beta provides a measure of the volatility in price of an asset (e.g., a stock) in relation to the broader market or index of the market. Forbes suggested that the reproductive returns from individual brood structures (e.g., mean fledging success in a given year) could be usefully equated to an individual asset, and that mean population reproductive success could be equated to the market as a whole. [16]

There is other study conducted on 1995 by Alex Kacelnik, Peter A. Cotton, Liam Stirling, and Jonathan Wright [3] which use evolutionary game theory to study Food Allocation among Nestling Starlings, drawing attention on Sibling Competition and the Scope of Parental Choice. Chick feeding in birds is often viewed as a prime example of evolutionary conflict. This is because the nestlings may benefit by inducing the parent to invest more in the current brood compared to future ones. In addition, each nestling should benefit by obtaining a greater fraction of the total brood provision than would be optimal for the parent. Current theory suggests that at evolutionary equilibrium, the intensity of signalling (i.e. begging) by the chicks should allow the parents to identify each chick's needs and to allocate more food to the one that offers the steepest marginal fitness gain per unit of parental resources (Godfray 1995). However, this study does not use any cooperative game concept.

1.2.2 Study on Cooperative Game in Biological Field

H. Peyton Young said in an interview written in a book titled **Game Theory: 5 Questions** edited by Vincent F. Hendricks and Pelle Guldborg Hansen [4] that one the most neglected topics and/or contributions in late 20^{th} century game theory is cooperative game theory. One reason, according to him, is that the topics in economics where game theory made its earliest inroads now seem particularly well-suited to the noncooperative approach. Another reason is because its practical applications have not been widely recognized even though generally, cooperative game theory is relevant to any situation where scarce resources are to be allocated fairly among a group of claimants. The last description is what we know as the concept of fair allocation.

Young mentioned one example of a fair allocation concept in biology or medicine study is when doctor consider which transplant patient should be first in line for the next kidney. He said that fairness must be judged in the context of the problem at hand, where criteria for allotting transplant organs may be quite different from criteria that pertain to the allocation of legislative seats, and neither may be relevant to the allocation of offices in the workplace or dormitory rooms at college. What is fair in one society, said Young, may not be deemed fair in another, because peoples expectations are conditioned by precedent, and precedents accumulate through the vagaries of history.

Unlike evolutionary game theory which was originally inspired by biological applications and thus has broad applications in the field of biology, cooperative game theory has more practical applications in the field of economics where topics of fairly sharing costs or dividing profits are already familiar to the readers. Only lately in 21^{st} century, there are some papers which address practical applications of cooperative game theory in biology, nevertheless none of them yet use the concept to study food allocation problem in bird broods.

A study conducted in 2007 by Claus-Jochen Haake, Akemi Kashiwada, and Francis Edward Su [5] on phylogenetic trees was using one of the cooperative game theory solution concept: the Shapley value. Interestingly, the study also suggests a biological interpretation behind the concept. The idea is, every weighted tree corresponds naturally to a cooperative game that is called a tree game, which assigns to each subset of leaves the sum of the weights of the minimal subtree spanned by those leaves. Here the leaves represent the species, and this assignment captures the diversity present in the coalition of species considered. This study also includes a brief discussion of the core of the tree game.

Aside from its use in phylogenetic trees, another use of cooperative game in earlier biological literature are given by coalition game studies (i.e. cooperative plant breeding games, paper by Eran Binenbaum and Phil Pardey, 2005 [6]; and cooperative bio-economic management of high seas fisheries problem, paper by Pedro Pintassilgo, 2002 [7]) and on building a classification model for disease classification study (i.e. leukimia, paper by Atefeh Torkamana, Nasrollah Moghaddam Charkarib, and Mahnaz Aghaeipourc, 2011 [8]). However, these studies might have minor connection on our topic of interest.

Chapter 2

Basics

To be able to model the problem of blackbird broods fair allocation, we would need to explain which methods or techniques we use in order to derive our results. We also need to describe the brood datasets that we use for the experiments, and how we will use them. Here in this chapter, we will start by giving an example of fair allocation in cooperative games, and continue by introducing some of the solution concepts such as the Shapley value and the Nucleolus. We then will explain some techniques we use in fitting the game from the known solutions. Finally, we will address the question of how we can convert the large sets of brood data to be interpreted within coalition concepts in a cooperative game context.

2.1 Fair Allocation in Game Theory

Below we describe some examples that draw attention to the earlier study of fair allocation and how these examples are connected in terms of finding one single solution that covers desirable properties of fair allocation problem using an approach from game theory. Note that almost all examples and definitions in this section are taken from the lecture notes on game theory by Frank Thuijsman [9].

2.1.1 A Bankruptcy Problem from the Talmud

"If a man who was married to three wives died and the kethubah of one was 100 zuz, of the other 200 zuz, and of the third 300 zuz, and the estate was worth only 100 zuz, then the sum is divided equally. If the estate was worth 200 zuz then the claimant of the 100 zuz receives 50 zuz and the claimants respectively of the 200 and the 300 zuz receive each 75 zuz. If the estate was worth 300 zuz then the claimant of the 100 zuz receives 50 zuz and the claimant of the 200 zuz receives 100 zuz while the claimant of the 300 zuz receives 150 zuz. Similarly if three persons contributed to a joint fund and they had made a loss or a profit then they share in the same manner." -Rabbi Nathan in Kethuboth, Fol. 93a, Babylonian Talmud, I. Epstein, ed., 1935

Before the field of Game Theory was established in the second half of twentieth century, the above Mishna has been stated long time ago in the books that hold Jewish religious and legal decisions: the Babylonian Talmud. The last sentence of the Mishna above has been specifically addressed by two mathematical game theorists Robert J. Aumann and Michael Maschler from the Hebrew University of Jerusalem. They studied this fair allocation problem (later known as the Bankruptcy Problem) using a game theory point of view [8]. They found that all informations from the Mishna (extracted into numbers in a table 2.1) corresponds exactly to the nucleolus of cooperative games that are related to the problems of sharing respectively 100, 200, and 300 among the three widows. In other words, one rule, involving the nucleolus from cooperative game theory (which we will introduce later in the next section), is able to explain the solutions given in the Mishna.

The story is popularly known as A Bankruptcy Problem from the Talmud and has been broadly used to describe a fair allocation problem within game theoretical context. Such a surprise that the study about nucleolus itself is introduced in 1969 by David Schmeidler, while the Mishna already 'use' it as a solution concept of fair allocation problem without 'studying' game theory yet. Note that we will define this nucleolus as well as other terms used in cooperative game later in the next section to be able to understand the whole concept, while now we are focusing on getting the story first in mind.

Table 2.1: The Talmud table

	Estate			min(Claim,Estate)		
Claim	100	200	300	100	200	300
A: 100	33.33	50	50	100	100	100
B: 200	33.33	75	100	100	200	200
C: 300	33.33	75	150	100	200	300

2.1.2 Contested Garment Principle

A similar example which has been found to be consistent with the Talmud table above [9] is the so-called contested garment problem, described as follows: "two persons hold a garment, one claims it all while the other claims a half. The first one is awarded 3/4, the other gets 1/4." An easy way to find the solution is by visualizing it into a rectangle, in which person A claims the full rectangle and person B claims only the right-side half of it. What can we see? That both persons are fighting over this right-side only. Person B will not argue if person A receives the whole left-side half. Thus according to the Mishna, this right-side that both claims to have, should be divided equally. Thus it follows that A gets 1/2 (the whole left-side half) plus 1/4 (half of a half), makes it in total of 3/4 for A, while B receives 1/4.

Given the Talmud problems, we will notice that the amounts given to the widows according to the Talmud table are all supported by this contested garment (CG)-priciple. To put it more precisely, for any two women, the division of their joint amount among the two of them is the CG-solution of this two-person problem [9]. In other words, the numbers in the Talmud table are consistent with the CG-principle.

Moreover, Aumann and Maschler in their paper [10] have shown that for any bankruptcy problem with any number of claimants, there is only one solution (among many solutions) that is consistent with the CG-principle. So if the Mishna says that other amounts have to be shared in the same manner, then we have to find a solution which is CG-consistent. Thus, they conclude, to share an amount of 250 among our three widows A, B, C claiming 100, 200 and 300 respectively, we have to find a solution (a, b, c) with properties:

1. a + b + c = 250;
2. for A and B the CG-solution for sharing a + b is (a, b);
3. for A and C the CG-solution for sharing a + c is (a, c);
4. for B and C the CG-solution for sharing b + c is (b, c).

2.1.3 Seeing the Problem as Coalition Procedures

"Samuel says that the Mishna assumes that the claimants have power over each other. More specific: The third woman can say to the second: 'You pay the first.' The second woman can say to the first: You wanted 100? Take 50 and leave.'" -a statement taken from the Talmud [9]

Since it is quite puzzling to figure out what are the right numbers according to the CG-principle, fortunately Aumann and Maschler have provided us a sequential (coalitional) procedure to solve any bankruptcy problem for the CG-consistent solution, which can be retraced to a statement in the Jerusalem Talmud above.

The statement refers to the cases where there is either 200 or 300 to share. Widows B and C are acting as a coalition claiming 500 against widow A who claims 100. Under these circumstances the CG-solution for sharing 200 would yield 150 for the coalition and 50 for widow A. When, next, widows B and C share 150 according to the CG-principle, then each of them gets 75. In a similar way one can derive the solution (50, 100, 150) for the 300 case. However, if we were to use this procedure for the 100 case, then again widow A receives 50, but B and C are getting 50 together and would share it half-half. Hence the result would be (50, 25, 25) and A who is claiming the least, is getting the most. This is clearly not CG-consistent, because the total of 75 for A and B should also have been shared equally.

Therefore, one property to be taking care of in finding a CG-solution for any bankruptcy problem is: for any two claimants, the one claiming more will never get less than the one claiming less. Therefore we have to be careful in the coalitional procedure, making sure that the members of the coalition are not going to get less than the one who is leaving the coalition with lower claim.

Another property that also has to be taken into account in the procedure is:

A claimant with a lower claim should never lose more than a claimant with a higher claim. In other words, the general procedure for finding a CG-consistent solution for a bankruptcy problem is: **small claimants get less and lose less than big claimants.** It has been shown by Aumann and Maschler that this coalitional procedure always yields the CG-consistent solution. They also show that this rule is self-dual, i.e. losses and gains are treated in precisely the same way. One thing to be kept in mind is, we have to keep track that the individual claimant is neither gaining too much, nor losing too much.

2.2 Modeling

In order to be able to understand the whole concept of cooperative game on which this fair allocation problem is based, we will provide some terminology and definitions ranging from the basic such as coalition, until the solution concepts like Shapley Value and nucleolus.

2.2.1 Cooperative Game

An n-person cooperative game is defined by the set of players N={1,2,...,n} and a function v which associates a non-negative real number $v(S)$ to every subset S of the grand coalition N. This $v(S)$ expresses the value/worth of the coalition, which is the amount that coalition S can achieve on its own effort without cooperating with those who are not in S.

As an example, below we show a 3-person cooperative game (person 1, 2, and 3 as players) with its possible coalitions.

Table 2.2: 3-person cooperative game

S	$\emptyset$	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
$v(S)$	0	1	3	4	4	5	8	10

Question is, given the table above, how to share the worth 10 of the grand coalition among the three of the players, 'fairly'? By examining their value alone, we can easily notice that player 2 is stronger (has higher value) than player 1 while player 3 is the strongest of all; yet it would not be that easy to assign the right allocation for each player.

Taking the Talmud table as an example, we will have three different 3-players cooperative games of sharing 100, 200, and 300 worth of coalition; where in each game the players' individual claims are 100, 200, and 300 respectively. How to put this value into such a coalition table? For this specific Talmud problem, Aumann and Maschler [10] identified the game by defining the worth of S as the amount that remains if the widow(s) not in S receive the claim first, leaving the rest for the others. Note that no one can receives more than the estate. Doing so, as stated in [9], we arrive at the following games:

Table 2.3: (100 | 100,200,300) game

S	$\emptyset$	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
$v_1(S)$	0	0	0	0	0	0	0	100

Table 2.4: (200 | 100,200,300) game

S	$\emptyset$	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
$v_2(S)$	0	0	0	0	0	0	100	200

Table 2.5: (300 | 100,200,300) game

S	$\emptyset$	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
$v_3(S)$	0	0	0	0	0	100	200	300

In the first table, Table 2.4, each widow is claiming at least 100; which is the same as what is available in the grand coalition. Thus, players outside S are leaving nothing for players in S. This makes all the amounts become zero except for the grand coalition. Another thing to be concerned is, if the widow(s) not in S claim more than the available amount, the worth of S would also then be zero.

We also know from Table 2.4 that $v_2(\{2,3\})$=100 since it is the estate that remains after subtracting the claim of widow 1 which is not in {2,3}. In other words, player 1, not being in the coalition {2,3} would leave only 100 of the available 200 to coalition {2,3}. The same explanation works for Table 2.5.

2.2.2 Solutions in Cooperative Game

Before we explain more on how to divide the value for all players in such a fair manner, we shall describe how each solution concept is built in detail. Using the definitions from [9], a solution for a cooperative game is a method for sharing the value of the grand coalition $v(N)$ among the individual players. Since $v(N)$ is achieved from cooperation of n players in which for each smaller coalition S, one might want to consider $v(S)$: the individual contributions of each player in establishing $v(N)$. Solving problem on finding the right share (allocation) for their joint profit or loss is the same as finding the solution that is fairly acceptable for such cooperative game.

It is obvious that each coalition S would prefer to get a share of at least $v(S)$. Unfortunately it might not always be possible. Here we denote an allocation of $v(N)$ to player $1, ..., n$ by $x = (x_1, ..., x_n)$, where player i receives x_i and $x_1 + ... + x_n = v(N)$. Allocations in which $x_i \geq v(\{i\})$ for each player i are defined as **individually rational**, while allocations with $x(S) = \sum_{i \in S} x_i \geq v(S)$ for each coalition S are called **coalitionally rational**. [17]

For any cooperative game (N, v), we call the set of coalitionally rational allocations as the core of a game, and denote it by $C(N, v)$.

Since there may be a lot of possible solutions for the allocations, we focus on the solution concepts which will give a one-point solution. Thus, the Shapley value, the Utopia value, and the Nucleolus came up as they always give a unique allocation for any cooperative game. As one-point solution concepts, the three concepts have some special properties which will be explained separately below.

Solution 1: Shapley Value

Coined by Lloyd Shapley (1953), this one-point solution concept introduced in his paper has some desirable properties called efficiency, anonimity, dummy, and addivity. A solution is **efficient** if it assigns to every game an allocation in such a way that the sum of every marginal contribution of each player will be equal to the value of the grand coalition.

The **anonimity** property is when the actual solution does not depend on the names of the players involved. It would mean that if we switch the role between who will become player 1 and who will be player 2, we want our solution to give both players as much as it would give them when they do not switch position.

If there is a player who does not contribute in any profit or loss of all coalitions he is involved, i.e. always contribute the same amount $v(\{i\})$ to any coalition, then he should receive that same amount $v(\{i\})$. This kind of player is called a **dummy player**. Player i is dummy if, for any coalition S not containing i, we have $v(S \cup \{i\}) = v(S) + v(\{i\})$.

The last one is the **addivity** property. A solution concept has this property if for any two games (N, v) and (N, w), the solution of $(N, v + w)$ is the sum of the solutions of (N, v) and (N, w). To understand this property, we shall look at two games [9] below as an example:

Table 2.6: 3-person game (N, v) and (N, w)

S	$\emptyset$	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
$v(S)$	0	3	1	4	4	8	5	10

S	$\emptyset$	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
$w(S)$	0	1	3	4	6	5	7	10

The additivity property applies if the solution of $(N, v + w)$ is the sum of the separate solutions of (N, v) and (N, w).

Table 2.7: 3-person game $(N, v + w)$

S	$\emptyset$	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
$(v + w)(S)$	0	4	4	8	10	13	12	20

Shapley (1953) in his paper titled "**A value for n-person games**" [23] provides a simple procedure to illustrate how to divide the coalition value fairly. According to him, the above 3-person game (N, v) can be illustrates as the following: Firstly, assume that player 1 enters the room and receives $v(\{1\})$, followed by player 2 (joining player 1 in the same room) which accordingly receives the marginal contribution $v(\{1,2\})-v(\{1\})$, and so on, until finally the last player n joins them altogether and receives $v(\{1,2,...,n\})-v(\{1,2,...,n-1\})$. Doing these value calculations **for every possible order** in which players enter the room, then taking the average values, will give us the unique/one-point Shapley solution (i.e. The Shapley value).

This Shapley value procedure gives each player i the average of its marginal contributions as calculated in the Shapley Value Formula (SVF) below. Expression $s!(n-s-1)!$ reflects the number of different orders where the first s players get together one by one, then player i joins in, and finally the remaining $(n-s-1)$ players join one by one as well.

$$\phi_i(N,v)=\sum_{S\subset N\setminus\{i\}} P_{n,s}(v(S\cup\{i\})-v(S)) \tag{2.1}$$

$$=\sum_{S\subset N\setminus\{i\}} \frac{1}{n}\frac{1}{\binom{n-1}{s}}(v(S\cup\{i\})-v(S)) \tag{2.2}$$

$$=\sum_{S\subset N\setminus\{i\}} \frac{s!(n-s-1)!}{n!}(v(S\cup\{i\})-v(S)) \tag{2.3}$$

The SVF above is based on this procedure: for player i a coalition S is chosen at random by firstly its cardinality, 0, 1, ..., $n-1$, say s is chosen (in which each cardinality has equal probability $\frac{1}{n}$), then among coalitions with cardinality s, one is chosen $\left(\text{each with the same probability } \frac{1}{\binom{n-1}{s}}\right)$.

This formula can be used as follows: player i receives $v(S\cup i)-v(S)$ $s!(n-s-1)!$ times, i.e. there are $s!(n-s-1)!$ orders where player i enters the room where coalition S is already present.

To be clear about the procedure of calculating the Shapley value, we will take the 3-person game in the previous Table 2.2 as an example. Below is the table where we list every possible order of every player's marginal contribution to the grouping of coalition (there would be six possible orders for a 3-person game in Table 2.2). If for example players group in the order of 2-3-1, then player 2's marginal contribution is $v(\{2\})-v(\emptyset)$=3-0=3, for player 3 the marginal contribution is $v(\{2,3\})-v(\{2\})$=8-3=5 and for player 1 it is $v(\{1,2,3\})-v(\{2,3\})$=10-8=2. We also calculate the average marginal contribution for each player (i.e. the Shapley-value): ϕ=$\left(\frac{8}{6},\frac{23}{6},\frac{29}{6}\right)$.

Table 2.8: Shapley value of the 3-person cooperative game

Possible orders	1	2	3
1-2-3	1	3	6
1-3-2	1	5	4
2-1-3	1	3	6
2-3-1	2	3	5
3-1-2	1	5	4
3-2-1	2	4	4
column sums	8	23	29
ϕ	$\frac{8}{6}$	$\frac{23}{6}$	$\frac{29}{6}$

Solution 2: Utopia Value

We now define the Utopia value of a cooperative n-person game, a one-point solution concept that is a part of a solution concept introduced by Stef Tijs (1981) [19] called the τ value. Let N be the set of all players and $v(N)$ be the value of the grand coalition. Given nonempty coalition $S \subseteq N$ and player $i \in N$, let $b_i := v(N) - v(N \setminus i)$ be the utopia vector of player i, which expresses the marginal contribution of player i to the grand coalition.

This utopia vector $\mathbf{b}$ gives an intuitive upper limit to what a player may expect to obtain from participating in the game. Player i would like to receive as much as his upper value, and he cannot hope for more than this value. Generally, every player will end up getting less than his utopia value, because for all interesting games, $v(N) \leq b_1 + ... + b_n$. If player i can get more than its utopia vector b_i, the other players might consider to throw i from the coalition as they would be better off without i [21]. Therefore in a core allocation, no player can ever get a payoff that exceeds this upper value.

However, the utopia vector $\mathbf{b}$ may not be efficient: summing up all the marginal contributions $\sum\limits_{i \in S} b_i$ of every player $i \in N$ may not be equal to the value of the grand coalition $v(N)$. To satisfy the efficiency property, the value that is distributed to the players will be divided by the sum of all utopia vectors for all players in N. Thus we define the Utopia value in our procedure as follows:

$$UV(N, v) = \Big(\frac{v(N)}{\sum\limits_{i \in N} b_i}\Big)\mathbf{b} \tag{2.4}$$

This utopia value is Shapley value-like, as its solution also has the additivity property. It also considers the marginal contributions, like the Shapley value. However, the utopia value takes only the marginal contribution regarding the grand coalition into account, while the Shapley value also takes the marginal contribution of all coalitions into account. Therefore this solution is simpler than the Shapley value solution. [20]

Solution 3: Nucleolus

David Schmeidler (1969) in his paper "**The nucleolus of a characteristic function form game**" [24] introduces the nucleolus as an alternative solution for cooperative games. The nucleolus is the set of individually rational allocations that lexicographically minimizes the excess/dissatisfaction of all coalitions. Unlike the Shapley value, the nucleolus is a solution of a minimization problem [11]. It is unique but exists only when these individually rational allocations exist; which in this case, if $v(N) \geq \sum_{i \in N} v(i)$. By definition, nucleolus will be in the core of the game whenever the core is non-empty.

Instead of applying a general axiomatization of fairness to a value function defined on the set of all characteristic functions, we look at a fixed characteristic function v, and try to find an allocation $x = (x_1, ..., x_n)$ that minimizes the worst inequity. That is, we ask each coalition S how dissatisfied it is with the proposed allocation x and we try to minimize the maximum dissatisfaction.

X is the nucleolus if and only if for all other allocations Z and all coalitions T that are better off with Z (i.e. $\sum\limits_{i \in T} z_i \geq \sum\limits_{i \in T} x_i$), there is a coalition T' that is better off with X, and x-dissatisfaction of T' is at least as large as the one of T [25].

There is an intuitive procedure to find this nucleolus [9]. If the core is non-empty, what we do is increasing the worth of all coalitions simultaneously, by the same amount, except for the empty set $\emptyset$ and the grand coalition N. This operation would make the new core become much smaller as we continue increasing until a further increment would result in an empty new core. If in the end a single point remains, then we got the nucleolus. Otherwise, if there are two or more coalitions remain but we cannot increase any further without creating an empty core (thus there are conflicting constraints), we then stop increasing the worth of these conflicting coalitions and keep on increasing the worth of every other coalition. We proceed this way until finally, we have a core consisting of just one point, which is the nucleolus.

On the other hand, starting with an empty core, we simply decrease the worth of all coalitions simultaneously, again except for the emptyset and the grand coalition, until we arrive at a game with a non-empty core. From that moment onwards, if we did not find a single point core, then we continue by increasing the worth of the non-conflicting coalitions, until again some conditions are on the verge of conflicting each other. The increase of the worth of conflicting coalitions is put on hold and we continue with the non-conflicting ones till a single point core remains: the nucleolus.

The nucleolus is one of the allocations that minimizes the maximal excess, i.e. if for a game (N,v) an allocation x=$(x_1,\ldots,x_n)$ is being considered as a solution, then one might wish to measure the level of each coalition's dissatisfaction within the possible solution. The difference between $v(S)$ and $x(S)$:=$\sum_{i \in S} x_i$ is taken as a measure of the dissatisfaction for each coalition S, and is called the **excess** $e(S,x)$.

$$e(S,x) = v(S) - x(S) \tag{2.5}$$

Note that each coalition would prefer a solution with the smallest excess as possible.

The problem of minimizing the maximum of a collection of linear functions subject to a linear constraint is easily converted to a linear programming problem and can thus be solved by the simplex method, for example. After this is done, one may have to solve a second linear programming problem to minimize the next largest excess, and so on. However, it is beyond the scope of this thesis.

In our procedure that will be explained later in the next subsection, we apply the **Prenucleolus** solution, which is preferred by the math-oriented game theorists, instead of the nucleolus which is preferred by the game theorists. In most cases, prenucleolus and nucleolus are considered the same. The difference with nucleolus is that there is no assumption of individually rational allocations when prenucleolus is considered. In the nucleolus, we only consider the individually rational allocations by letting contribution for player i, $x_i \geq v(\{i\})$. While the prenucleolus only consider efficient allocations by looking at the contribution $x \in \mathbb{R}^N$ for which $\sum\limits_{i \in N} x_i = v(N)$.

Nucleolus and prenucleolus are overlapping in a class of nonnegative games where this inequality $v(T) - \sum\limits_{i \in T} v(i) \leq v(S) - \sum\limits_{i \in S} v(i)$ holds [12] for the two coalitions $T \subseteq S$. In our context, this inequality seems to hold. Therefore in this thesis, we may say the nucleolus and prenucleolus are the same. To avoid confusion, from now on we will call our solution concept as the Prenucleolus.

2.3 The Fitting Techniques

In the previous section, we already have three solution concepts in cooperative game, namely the Shapley value, the Utopia value, and the Prenucleolus. To be able to reconstruct the game using the solution concepts given and the brood data which will be explained in section **2.4**, we need some techniques to be implemented in our procedure of finding the best fitted game. Here we describe how can we make the game from the brood data that will be translated into solutions, assuming this game is solved by the three solution concepts mentioned earlier. To determine how good this game fits the solutions, we will also introduce an error measurement in this section.

2.3.1 Techniques on Shapley Value

As we mentioned before, Jean Derks [18] described three procedures in order to find a fit for the brood data, making use of some mathematical concepts such as Balanced Contributions and Unanimity games. Below we will explain and examine how these approaches look like, then choose one of them to be implemented in our brood data.

Firstly, let N be a fixed, finite set of players, and let Ω be a set of subsets of N. Suppose for each coalition $S \in \Omega$ there is a payoff vector $x^S = (x_i^S)_{i \in S}$, expressing the profits of the players in S when they decide to cooperate. In other words, this x^S is a set of solutions of a cooperative game. We address the problem of how to find this corresponding cooperative game. Assuming that the payoffs follow the Shapley value distribution, the following approaches are considered.

Balanced Contributions

One approach is to assume some kind of fairness or balancedness that also holds for the Shapley value [26]. Consider a so-called payoff system $Z = (z_i^S)_{S \subset N, i \in S}$. Z is said to be balanced if equation 3.2 below holds for all coalitions S and players $i, j \in S$.

$$z_i^S - z_i^{S \setminus \{j\}} = z_j^S - z_j^{S \setminus \{i\}} \tag{2.6}$$

The intuition behind this property is as follows: the amount $z_i^S - z_i^{S \setminus \{j\}}$ is the loss player i experiences when player j decides to leave the coalition S; so, Z is balanced if each two players in any coalition attain the same loss when the other decide to leave the coalition.

Naturally, we may not be able to extend the above introduced collection $X = (x^S)_{S \in \Omega}$ into a balanced payoff system Z but it is interesting to investigate conditions on Ω ensuring the existence of balanced extentions. This is, however, not pursued in this thesis.

Game-Fitting Procedure

For this procedural approach, we define a **unanimity game** for coalition S

$$U_S(T) = \begin{cases} 1 & \text{if } S \subseteq T,\, T \subseteq N \\ 0 & \text{otherwise} \end{cases} \tag{2.7}$$

These unanimity games $\mathbf{U}$ form a basis for the set of games that we will use in our procedure that will be explained in the next approach. In this unanimity game, all players in S should be present in order to make the coalition T to be powerful. The Shapley value for this unanimity game is given by:

$$\phi(U_S) = \begin{cases} \dfrac{1}{|S|} & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases} \tag{2.8}$$

Note that here we divide 1 with the number of players in S. We can see that the payoff vector $\phi(U_S)$ is now efficient.

Suppose that we fit the data and arrive at a game (N, v). How can we improve the upcoming procedural approach in some sense 'better'. From the existing game (N, v), we only consider new games $(N, v + \alpha u)$ with a scalar α, and $u \in U$, a finite, fixed set of games. As already mentioned before, here we take $\mathbf{U}$ to be the set of unanimity games. For each unanimity game (N, u_T) we first compute the weight α_T for which the error is minimum, i.e. satisfying this equation:

$$\mathcal{E}(v + \alpha_T u_T) = min_\alpha \mathcal{E}(v + \alpha u_T). \tag{2.9}$$

Then, choose coalition $\hat{T}$ with:

$$\mathcal{E}(v + \alpha_{\hat{T}} u_{\hat{T}}) = min_T \mathcal{E}(v + \alpha_T u_T), \tag{2.10}$$

and let v' be $v + \alpha_{\hat{T}} u_{\hat{T}}$.

In other words, we only change the existing game (N, v) into a new game (N, v') in the direction of one unanimity game.

We will explain more on this error $\mathcal{E}$ in the following subsection.

Error Measurement

After we compute the new game (N, v') using the procedure above, we want to see if this game is close to the game we are looking for. Firstly, we will define two measures of 'closeness'. The first measurement e_D is where we compare each brood data points with each solution points of the Shapley value procedure in order to measure how well the game solutions fit the brood data we have. The second one is $\mathcal{E}(v)$, in which we measure how well the brood data fits the solutions of the new game we found from the fitting procedure. Assume that the importance weights $I_{S \in \Omega}$, are provided from the brood data; the smaller the error $\mathcal{E}(v)$, the closer we would like the value of the coalitions $v(S)$ to our payoff vectors x^S. The following error measure fulfills this property:

$$\mathcal{E}(v) = \frac{\sum\limits_{S \in \Omega} I_S \frac{\left| \sum\limits_{i \in S} x_i^S - v(S) \right|}{\sum\limits_{i \in S} x_i^S}}{\sum\limits_{S \in \Omega} I_S} \tag{2.11}$$

Note that we firstly take the absolute error in order to avoid negative results in the summation, then we take the mean by dividing this absolute error with the total of our payoff vectors. We then take the relative error by multiplying the absolute error with the importance weights of every possible coalitions S in Ω. Finally we divide it with the sum of the importance weights, to get the total error measurement.

For the error measure e_D, we calculate the difference between the new game $(y_{i \in S}^S)_{S \in \Omega}$ with our payoff vector. This y_i^S is defined as the Shapley value solution of the subgame v with weight w (see next subsection for this weighted version), in which only players in S are taken into account. The error e_D is mathematically define as follows:

$$e_D = \frac{\sum\limits_{S \in \Omega} I_S \sqrt{\sum\limits_{i \in S} \left(x_i^S - y_i^S\right)^2}}{\sum\limits_{S \in \Omega} I_S} \tag{2.12}$$

We firstly take the square of the difference in order to avoid negative values, sum it over all players in S, then take the square root before multiplying it with the sum of the importance. In the end, we also divide the result with the sum of all importance weights.

Here we take $y_i^S = \dfrac{99}{100} x_i^S$ to be the new game in order to get the 1% average of error as a value that we would like to achieve.

The error $\mathcal{E}(v)$ is smallest among all games derived from v by adding a weighted unanimity game. By repeating the game-fitting procedure above, for example by starting with the zero game, we may arrive at a game with error below a given level, or when the error change is below a specified level (for example 0.0001 of the desired error level). This error $\mathcal{E}(v)$ is our focus on the experiments later, as the smallest $\mathcal{E}(v)$ would bring us closer to the best fitted game in which the value of the coalitions $v(S)$ is close to our payoff vectors x^S.

Weighted Shapley Value

If the resulting error is still high, and in order to capture the problem that different players evaluate the payoff differently, instead of using the unanimity game in the above repeated procedure where the payoffs to the players are treated equally, we may consider the weighted Shapley value approach [27]. By assuming that there are (unknown) weights $w_i, i \in N$, such that the payoff of one unit is actually worth w_i to player i, then we should consider the set of allocations:

$$x^w = \left(\frac{1}{w_i} x_i^S\right)_{i \in S, S \in \Omega} \tag{2.13}$$

in the above approach.

Let $\mathcal{E}(v)$ denotes the error and v^w the game we get if we apply the repeated procedure on x^w. It can be proven that $\mathcal{E}(v)$ equals the error on v^w, but instead of the Shapley value the weighted Shapley value is chosen with weight system w_i.

In computing weights with a satisfactory error we may follow the same idea as before. Let $W = \{w^1, ..., w^K\}$ be a set of weight systems:
for each weight system w^k compute the weight α^K, α:=0 to 1, for which

$$\mathcal{E}(w + \alpha_k w^k) = min_\alpha \mathcal{E}(w + \alpha w^k). \tag{2.14}$$

Then, choose index $\hat{k}$ with

$$\mathcal{E}(w + \alpha_{\hat{k}} w^{\hat{k}}) = min_T \mathcal{E}(w + \alpha_k w^k). \tag{2.15}$$

and let w' be $w + \alpha_{\hat{k}} w^{\hat{k}}$.

Again, by repeating this procedure, for example by starting with weights $w_i = 1, i \in N$, we may arrive at a game with desirable error.

Naturally, the weight systems in W may be chosen such that the needed computations are effectively implemented. However, no such W is known, and therefore we propose the following procedure, with α a given positive number.

As long as there is a $w^k \in W$ such that $\mathcal{E}(v) > \mathcal{E}(w + \alpha w^k)$ then change v into $w + \alpha w^k$. If no improvements are observed then change α into a smaller number, then repeat this procedure. The repetition can be maintained as long as computation time is available or the desirable error is reached.

2.3.2 Other Solution Concepts as Comparison

In order to compare the performance of the Shapley value procedure in our brood data, a similar fitting procedure is applied on the weighted version of the other two approaches (i.e. the Utopia value and the Prenucleolus). Note that we will not explain the techniques for these two other solution concepts as the same approaches described for the Shapley value above are mimicked for both Utopia value and Prenucleolus.

2.4 The Data and The Game Translation

In this section we will describe what are the data we have and how can we translate the raw data using the model and the techniques explained in the previous sections, into something useful and insightful.

2.4.1 The Brood Data

The brood datasets that are used in this thesis have been collected by Professor Scott Forbes for about ten years during his research in wetlands near Winnipeg, Manitoba, Canada. It is defined in Forbes book [1] that the core brood/chicks are the nestlings that are hatched together on the first day of the nestling period; while nestlings that are hatched one or more days later are defined as the marginal brood/chicks. The blackbird parents' choice of hatching how many eggs in the first day could be based on their experiences on the previous hatching periods, or on their instinct of the weather and food condition near the nest. They will hatch each one of the marginal every one day after the core. Thus, having 2 core chicks and 3 marginals will make the hatching periods of 4 days in total (1 day for all the cores, 3 additional days for each marginal).

The raw data in Table 2.9 and Table 2.10 below enable us to find out how many core and marginal children that the blackbirds could have in one brood, as well as how many broods are available for the specific number of core and marginal chicks. We can also see how many chicks that are perished during one week of measurement (starting from day 1 to day 8). These large datasets have been compiled into good years and bad years period, which allow us to see whether there is any difference on how parents allocate the food for the core and marginal children during the good and the bad times.

Note that notation **#eggs** shows how many eggs are firstly available in total before the hatching period. These eggs might be removed, broken, or perish early during a hatching failure, thus this total number can sometimes be different with the number of hatchlings. Notation **c** in day 1, denotes the number of core eggs (eggs that are hatched in day 1) inside one brood; while notation **m** denotes the number of marginal eggs (eggs that are left/not yet hatched in day 1) in one brood. Notation **#br** shows how many broods that are available in total. In day 8, **c** and **m** denote number of core and marginal chicks which survive within one week, while **m1**, **m2**, **m3** denote the number of marginals that are hatched on the first, the second, and on the third day after the core. The **total** in day 8 shows how many chicks that are continue living after one week of feeding.

Table 2.9: The good years raw data

	day 1			day 8					
#eggs	c	m	#br	c	m	m1	m2	m3	total
18	1	0	15	15	0	0	0	0	15
59	1	1	29	27	27	27	0	0	53
120	1	2	40	38	62	33	29	0	95
218	1	3	55	54	124	52	48	24	176
30	2	0	15	28	0	0	0	0	28
236	2	1	78	134	56	56	0	0	186
430	2	2	108	208	140	85	55	0	345
75	2	3	15	29	26	14	9	3	55
81	3	0	27	74	0	0	0	0	74
292	3	1	73	196	33	33	0	0	224
100	3	2	20	55	16	12	3	0	71
32	4	0	8	27	0	0	0	0	27
25	4	1	5	19	2	2	0	0	21

Table 2.10: The bad years raw data

	day 1			day 8					
#eggs	c	m	#br	c	m	m1	m2	m3	total
6	1	0	6	6	0	0	0	0	6
40	1	1	20	22	17	16	1	0	35
100	1	2	33	39	48	26	22	0	63
116	1	3	29	46	56	27	22	6	68
50	2	0	24	46	0	0	0	0	46
153	2	1	51	104	35	34	0	0	121
292	2	2	73	136	61	44	17	0	168
55	2	3	11	19	13	9	3	1	29
54	3	0	18	44	0	1	0	0	44
331	3	1	83	201	25	25	0	0	198
40	3	2	8	23	1	1	0	0	22
28	4	0	7	21	0	0	0	0	21
15	4	1	3	9	1	1	0	0	10

As we have seen in the above data, one brood can contain at maximum 4 core chicks, while on the other hand, it can also contain at maximum 3 marginals.

However, even though the total number of chicks is seven (i.e. there are seven players in the game), we do not have data with four core and three marginals at the same time. The largest brood we have consists of five chicks, either 2 core with 3 marginals, 3 core with 2 marginals, or 4 cores with 1 marginal. Note that from the story of blackbirds in Forbes's book [1], the parents will feed the chicks that beg louder, which usually are the core chicks. Therefore, in one way to calculate the Shapley value, we consider to assume that the feeding process will always start with the core chicks, while the marginals 'fight' over the remaining food after the cores are being fed. This assumption will be described later in the next chapter during calculations and experiments of our method.

In theory, we also cannot have only marginals without having the core, or having the third and/or the second marginal without having the first one. But this is happening in some of the brood data since there is a possibility that the egg is missing or being destroyed during the hatching period of the core chick, not to mention the chick that is directly dead after born, leaving only the marginals in the brood. Same case is also happening for the marginals. However, later in the next chapter we will see that our method excludes this kind of missing data from the calculations, and consider only the feasible coalitions.

As a result of being born on different days where the marginals are hatched on each day after the cores, we think our brood data have a specific property: there exist different weights between core and marginal chicks. This is because the core and the marginal chicks may value their food in different way. We predict that all the core chicks c will value their food in the same manner, since they are hatched on the same day (thus may be as strong as each other) and the parents consider them to be equally important to continue the family legacy. As a result, the weights' difference between the core chicks is very small or can be ignored. In other words, we assume that competition between the core chicks in one brood are not exist. However, there exist different weights between the core and the marginals, as well as between all the marginals, as the marginal m_i, i :=1 to 3, are born consecutively on i days after the core. Note that this weight is not a body-mass index but an additional number that represents how the chicks may value their food. This weight will be explained further in the next chapter.

2.4.2 Coalition Model in the Brood Data

To be able to build a coalition model for the brood food allocation data during the good and the bad years, we firstly we define an XY-brood game where X and Y denote the number of core and marginal children respectively. As input for the bird brood food allocation model, we use the average of the survival rate data **A** and the importance weights data **I** which shows how many times a specific type of brood (i.e. core and marginal coalition) appears in the game. Note that in Table 2.11 and Table 2.12 below, notations **c** and **m** of the **A** data represent the number of core chicks that are hatched in day 1, and the number of marginal chicks that are left (will be hatched consecutively in the next days). Notation **d8 val** represents the average number of all chicks (both core and marginals) which survive until one week of feeding (i.e. day 8). Notation **m av** shows the average of the marginal chicks' survival rate which we will use to fill the coalition value for the marginals later in our method, while each notations **c val, m$_1$ val, m$_2$ val**, and **m$_3$ val** show the survival rate of each chick, starting from the core, the first marginal, until the third marginal chick respectively.

Table 2.11: Average of the Survival Rate for XY-brood type during the Good years

A								
c	m	c val	m av	m_1 val	m_2 val	m_3 val	d8 val	**I**
1	0	1.000					1.000	16
1	1	0.931	0.931	0.931			1.862	29
1	2	0.950	0.838	0.892	0.784		2.626	40
1	3	0.982	0.765	0.963	0.889	0.453	3.287	55
2	0	0.933					1.867	15
2	1	0.859	0.747	0.747			2.465	78
2	2	0.963	0.688	0.794	0.514		3.234	108
2	3	0.967	0.578	0.933	0.600	0.200	3.667	15
3	0	0.914					2.741	27
3	1	0.895	0.465	0.465			3.150	73
3	2	0.917	0.400	0.632	0.158		3.539	20
4	0	0.844					3.375	8
4	1	0.950	0.400	0.400			4.200	5

Table 2.12: Average of the Survival Rate for XY-brood type during the Bad years

A								
c	m	c val	m av	m_1 val	m_2 val	m_3 val	d8 val	**I**
1	0	1.000					1.000	6
1	1	1.000	0.850	0.850			1.850	20
1	2	0.848	0.774	0.839	0.710		2.397	33
1	3	1.000	0.644	0.931	0.759	0.453	2.897	29
2	0	0.958					1.917	24
2	1	0.931	0.686	0.686			2.549	51
2	2	0.836	0.418	0.603	0.233		2.507	73
2	3	0.818	0.433	0.900	0.300	0.200	2.936	11
3	0	0.815					2.444	18
3	1	0.763	0.309	0.309			2.598	83
3	2	0.917	0.063	0.125	0.000		2.875	8
4	0	0.750					3.000	7
4	1	0.750	0.333	0.333			3.333	3

Having the coalitions S, what value can we choose to be the value of the coalition $v(S)$? Since we have the average of the survival rate **A** for each offsprings in every XY-brood, taking into account its importance weight **I** (i.e. how many times the XY-brood data occur), we can take these values as the value of the coalitions. But how to put the 'right' value into the 'right' coalition? Here we propose a coalition procedure which we adapt from the Shapley value procedure in the previous subsection by seeing the problem as a coalitional procedure as shown in the example given in subsection **2.1.3**.

1. Suppose we have an XY-brood game during either the good or the bad years, with maximum 7 players (i=1 to 7), consists of maximum 4 core (i=1 to 4) and maximum 3 marginal players (i:=5 to 7). Firstly we define all possible coalitions of (X+Y) players where there are X core players and Y marginals.

   For example, if we take the good years 21-brood data where there are two core players and one marginal player, it is possible to have coalitions of every single player, coalition between the core players, coalitions between each core with the marginal, and coalition of all the three players. In other words, for the good years 21-brood, the possible coalitions of the 3 players are: $\{1\}$, $\{2\}$, $\{5\}$, $\{1,2\}$, $\{1,5\}$, $\{2,5\}$, and $\{1,2,5\}$.

2. We then re-translate every possible coalition into the number of core and marginal players in a brood.

   For example, coalition $\{1\}$ and $\{2\}$ is when we only have one core player in a brood, without having any marginals; i.e. the 10-brood. Thus, for the single core player coalitions, we will consider the 10-brood game. Now we do the same translations for the other coalitions: consider 20-brood game for the coalition between the core players $\{1,2\}$, 11-brood game for the coalitions of each core player with the marginal (i.e. $\{1,5\}$ and $\{2,5\}$), and simply 21-brood game itself for the coalition of all three players $\{1,2,5\}$. As an exception, for the marginal player coalition $\{5\}$ in 21-brood data, we cannot take the 01-brood into account, since by definition, no marginal can be hatched before having the core hatched. Thus we do not need to consider this kind of brood in the procedure.

3. Now we continue by looking at Table 2.11 for the good years, and Table 2.12 for the bad years, in order to see the average of the survival rate in the corresponding XY-brood data, which we need to consider for each coalition.

   As an example, to fill in the coalition value $v(S)$ of the single core player coalitions $\{1\}$ and $\{2\}$, we take the average of the survival rate for this core chick in 10-brood, which is 1.000 (see **c val** on the table). For the marginal player coalition $\{5\}$ which is an exception, we choose to take the survival rate of the marginal on its first appearence in the brood datasets, which is the m_1 **val** value on table: 0.931. Note that this m_1 **val** is equal to the **m val** as we only have one marginal in 10-brood game. For the 20-brood, 11-brood, and 21-brood game, we take the corresponding total average number of the chicks' survival rate **d8 val**, which are 1.867, 1.862, and 2.465, respectively.

As a result, we then have this coalition table for the 21-brood game during the good years period:

Table 2.13: 21-brood game, good years

S	{1}	{2}	{5}	{1,2}	{1,5}	{2,5}	{1,2,5}
$v(S)$	1.000	1.000	0.931	1.867	1.862	1.862	2.465

Note that we remove the emptyset coalition ($\emptyset$) in the brood coalition table since in any case it would always take a zero value. Same procedure applies for every XY-brood game.

Chapter 3

Experiments

Before adapting the brood data into a program, we think it is better to get the feeling of how the brood data looks like; so that we know what is the best choice to implement this data into the program. Therefore in this chapter, firstly we will show how we do some calculations by hand on the Shapley value solutions for some XY-brood game in the good and the bad years data. After that, we will provide some results of the experiments using Matlab for the three solutions described in the previous chapter, namely the Shapley value, the Utopia value, and the Prenucleolus, using two different cooperation approaches called the standard approach and the restricted approach. We will also show how we translate the survival rate into some kind of utility functions and which translation gives the best fit with the smallest minimum error. We are using the good and the bad years data that have been compiled by Professor Scott Forbes, as well as random data for validation. Our focus is onto the Shapley value, but we will compare its results with the other two solution concepts.

3.1 Calculations by Hand

Here in this section, we will show that for some cases of the brood data in the good and the bad years period, it is possible to calculate the Shapley value solution by hand. However, the larger the data, the more we need a program which can easily find the solutions using the Shapley value solution concept, or the other cooperative game solution concepts in a quite short period of time.

3.1.1 Calculating Shapley Value of the Good Years Data

As we can see in the brood data for the good years (Table 2.11), there are data of 10-brood, 11-brood, 12-brood, 13-brood, 20-brood, 21-brood, 22-brood, 23-brood, 30-brood, 31-brood, 32-brood, 40-brood, and 41-brood. To get a feeling of these data we are having, we will calculate the Shapley value ϕ by hand for some of the smaller broods data, to see if we can get something interesting as a result.

There are two different approaches that we use in our procedure of calculating the payoff vectors x^S for the coalition S. The first one is the **standard approach**. Remember that we have at maximum seven players ($|N|$=7), consist of at maximum 4 core players (i=1 to 4) and at maximum 3 marginal players (i:=5 to 7). Thus, we may consider seven different places for each different positions of the players. In this standard approach, the core chicks can be placed in anywhere among the four first places while the marginals are placed consecutively in the three last places (in an increasing order). As the core may become the first, the second, the third, or the fourth player, it can be placed in any of the four first places. Thus, we need to consider the same survival rates for all these four possible places of the core chicks in each XY-brood game; while the importance weights **I** for the corresponding core chicks are divided equally among the four possible places. Note that whichever chick chooses the first place will be considered as the first core, and so on.

To be clear about this representation, we convert Table 2.11 of the good years data into a new table, Table 3.1, by defining an allocation x^S as the average of every chick's survival rate in the corresponding XY-brood game, taking into account the possible coalitions that can be made by all the players involved in the game. For example, if we consider the good years 21-brood game using the standard approach, we will have a set of possible coalitions which consists of coalition {1,2,5}, {1,3,5}, {1,4,5}, {2,3,5}, {2,4,5}, and {3,4,5} since the two core chicks can choose any of the four first places. Note that what we denote as possible coalitions here are the places where the chicks exist.

The Table 3.1 below will show how many players involved in each coalition of a specific XY-brood, what are the possible coalitions exist in a specific XY-brood, and what are the survival rates of each player involves in those specific coalitions. Note that the numbers i:=1 to 7 in the table denote the players, where i=1 to 4 are core players and i:=5 to 7 are marginal players.

Table 3.1: Survival rates of the players in the existing coalitions (Standard Approach, Good Years)

		x^S			
XY	Possible coalitions S	1 to 4	5	6	7
10	$\{1\}, \{2\}, \{3\}, \{4\}$	1.000	0	0	0
11	$\{1,5\}, \{2,5\}, \{3,5\}, \{4,5\}$	0.931	0.931	0	0
12	$\{1,5,6\}, \{2,5,6\}, \{3,5,6\}, \{4,5,6\}$	0.950	0.892	0.784	0
13	$\{i,5,6,7\}, \forall i$:=1 to 4	0.982	0.963	0.889	0.453
20	$\{i,j\}, \forall i,j$:=1 to 4, $i < j$	0.933	0	0	0
21	$\{i,j,5\}, \forall i,j$:=1 to 4, $i < j$	0.859	0.747	0	0
22	$\{i,j,5,6\}, \forall i,j$:=1 to 4, $i < j$	0.963	0.794	0.514	0
23	$\{i,j,5,6,7\}, \forall i,j$:=1 to 4, $i < j$	0.967	0.933	0.600	0.200
30	$\{i,j,k\}, \forall i,j,k$:=1 to 4, $i < j < k$	0.914	0	0	0
31	$\{i,j,k,5\}, \forall i,j,k$:=1 to 4, $i < j < k$	0.895	0.465	0	0
32	$\{i,j,k,5,6\}, \forall i,j,k$:=1 to 4, $i < j < k$	0.917	0.632	0.158	0
40	$\{1,2,3,4\}$	0.844	0	0	0
41	$\{1,2,3,4,5\}$	0.950	0.400	0	0

Note that in total we will have 54 coalitions if we consider the standard approach.

The second approach is the **restricted approach**. Here we restrict n core players, n=1 to 4, to always be in the first n-places, while the remaining $4 - n$ places that are not taken by the core chicks will have a zero value. For example, if we have a 21-brood where there are two core players and one marginal, then the two core players will always fill the first and the second places, while the third and the fourth places remain zero. For this restricted approach, below we will see how the average of the chicks' survival rates are also placed into the table together with the existing coalitions. Note that we will have only 13 coalitions if we consider this approach. This number is the same as the number of all existing XY-brood games.

Table 3.2: Survival rates of the players in the existing coalitions (Restricted Approach, Good Years)

		x^S						
XY-brood	S	1	2	3	4	5	6	7
10	{1}	1.000	0	0	0	0	0	0
11	{1,5}	0.931	0	0	0	0.931	0	0
12	{1,5,6}	0.950	0	0	0	0.892	0.784	0
13	{1,5,6,7}	0.982	0	0	0	0.963	0.889	0.453
20	{1,2}	0.933	0.933	0	0	0	0	0
21	{1,2,5}	0.859	0.859	0	0	0.747	0	0
22	{1,2,5,6}	0.963	0.963	0	0	0.794	0.514	0
23	{1,2,5,6,7}	0.967	0.967	0	0	0.933	0.600	0.200
30	{1,2,3}	0.914	0.914	0.914	0	0	0	0
31	{1,2,3,5}	0.895	0.895	0.895	0	0.465	0	0
32	{1,2,3,5,6}	0.917	0.917	0.917	0	0.632	0.158	0
40	{1,2,3,4}	0.844	0.844	0.844	0.844	0	0	0
41	{1,2,3,4,5}	0.950	0.950	0.950	0.950	0.400	0	0

After having the two tables above, we will explain two ways of calculating the Shapley value ϕ for 12-brood by hands. The first one is done for every possible orderings of the grand coalition, while the second one is by removing the orders that are assumed to be unfeasible before we start calculating the value. To understand how this calculation method works, we will denote a set of possible orders $\mathbf{P}$ as any possible orders of the chicks when they are being fed by the parents: starting from the firstly fed chick, until every chick in the corresponding XY-brood data is being fed. For example, order 1-5-6 in the 12-brood game means that the core chick is being fed at the first place, followed by the first and the second marginals consecutively.

As we have mentioned in the previous chapter,when blackbirds parents come to the nest bringing the foods for their chicks, whichever chick that begs harder will be fed first with usually the largest amount of food, and vice versa; chick that is being fed last will get just the remainder. As food given from parents is probably the only source of the chicks' nutritions, at least until they are able to fly and look for another source of food, this food is very important for them to survive. Therefore, we may logically assume that the chick's order of being fed will affect their survival rate. Using this assumption, it is possible to calculate the Shapley value by taking the average of the chicks' survival rate in the corresponding brood data to be interpreted as the amount of food the chicks are getting from their parents which will help them to survive. An average of 1.000 for a chick's survival rate could be translated as: the chick is getting 100% of food that it needs to survive.

In order to calculate the Shapley value by hand using the translation above, we define the following allocation procedure:

1. Consider the XY-brood game during either the good or the bad years period under the restricted approach. Make a coalition table for the XY-brood game using the coalition procedure described in subsection **2.4.2**.

   As an example, now we consider the good years 12-brood game. Using the procedure that are explained before in section **2.4.2** and looking at Table 2.11 for the chicks' average survival rate data during the good years, notice that we use the total sum of all chicks' survival rate in 12-brood to fill in the value of the grand coalition, while the marginals' average **m av** of the 11-brood and 12-brood are used to fill in the value of the coalition {5} and {6}, respectively. To fill in the value for coalition {1,5} and {1,6}, we use the sum of the survival rate for core chick in 11-brood with the **m av** of 11-brood and 12-brood respectively. Finally, the sum of m_1 and m_2 survival rate of the 12-brood is used to fill in the value for coalition {5,6}. Thus, we have a coalition table for the 12-brood game as follows:

   Table 3.3: 12-brood game, good years

S	{1}	{5}	{6}	{1,5}	{1,6}	{5,6}	{1,5,6}
v(S)	1.000	0.931	0.838	1.862	1.769	1.676	2.626

2. List every possible orders of the grand coalition that correspond to this XY-brood data.

3. In order to be able to fill in the 'right' value that every player will get according to their possible ordering, we adapt the same Shapley procedure as explained in the previous chapter. This way, we will divide the value of the grand coalition 'fairly' by considering the orders and the value that are 'claimed' by each coalition.

   As an example, from Table 3.3, we know that player 1 in coalition {1} is 'claiming' an average of 1 for its survival rate, while player 5 and 6 in coalition {5} and {6} are 'claiming' an average of 0.931 and 0.838, respectively. If we take into account order 1-5-6 of the players in the grand coalition, we will firstly allocate 1.000 for player 1; exactly the same amount as what it claims. To decide how much should player 5 gets, we look at Table 3.3 and see that 1.862 is the value of coalition {1,5}. Since we already give player 1 a value of 1.000, the remaining value of 0.862 will be the amount which is given to player 5. Keep in mind that the sum of every player's value needs to be equal to the value of the grand coalition. Since a total of 1.862 has already been given to player 1 and 5, player 6 will get the remainder of the grand coalition value; which is 0.764. Doing the same procedures to every possible orders, we will get a Shapley value calculation table as shown below. Note that notation ϕ denotes the Shapley value of each player involved in the grand coalition.

Table 3.4: Shapley value of 12-brood game, good years

Possible orders	Player			Total
P	1	5	6	
1-5-6	1.000	0.862	0.764	
1-6-5	1.000	0.857	0.769	
5-1-6	0.931	0.931	0.764	
5-6-1	0.950	0.931	0.745	
6-1-5	0.931	0.857	0.838	
6-5-1	0.950	0.838	0.838	
ϕ	0.960	0.879	0.786	2.626

4. Now we compare the values we got from observing the average survival rates of each chick in the corresponding XY-brood data, which we denote as *Observ.*, to the Shapley values we got from calculation. In order to get these observation values, we need to look at Table 3.2 (restricted approach) and find the average survival rate of each chick in the corresponding XY-brood.

   For example, the observation values for player 1, player 5, and player 6 in 12-brood game according to Table 3.2 are 0.950, 0.892, and 0.784, respectively. For easier comparison, we will add these observation values into the Shapley value calculation table we made in the previous step, resulting this table below:

Table 3.5: Possible orders for 12-brood game, good years

Possible orders	Player			Total
P	1	5	6	
1-5-6	1.000	0.862	0.764	
1-6-5	1.000	0.857	0.769	
5-1-6	0.931	0.931	0.764	
5-6-1	0.950	0.931	0.745	
6-1-5	0.931	0.857	0.838	
6-5-1	0.950	0.838	0.838	
ϕ	0.960	0.879	0.786	2.626
Observ.	0.950	0.892	0.784	2.626

5. Since there is almost no case in the blackbird broods where the marginal chicks are being fed before the core chicks, now we consider to leave out the unfeasible orders from the Shapley value calculation table and consider only the cases when the core chicks are being fed before the marginals. We now have a new Shapley value calculation table with a set of feasible orders **F** instead of possible ones. Below is the new Shapley value calculation table for the 12-brood game:

Table 3.6: Feasible orders for 12-brood game, good years

Feasible orders F	Player 1	 5	 6	Total
1-5-6	1.000	0.862	0.764	
1-6-5	1.000	0.857	0.769	
ϕ	1.000	0.8595	0.7665	2.626
Observ.	0.950	0.892	0.784	2.626

> Again, we compare the Shapley value ϕ with the observation value to see if parental favoritism exists in the case of a specific XY-brood data.

Notice that in the case of 12-brood, the Shapley values ϕ for the core chicks that we got in both cases are larger than the observation values. Thus there is no tendency of parents favoriting the core chicks according to this Shapley value solution in 12-brood. On the other hand, the Shapley value for the marginals are almost always larger in the observations rather than in the calculation; except for the second marginal in the case of taking all possible orders **P** into calculation. Thus we may say that in the good years 12-brood data, there is no indication of blackbirds parents playing favorites between the core and the marginal chicks.

Would it still be the case for every XY-brood game during the good years, or will the parents start favoriting the core chicks at some point? Note that the same way of calculations can also be applied for every brood game, especially the smaller ones (with not more than three or four players in one brood).

Here we provide another example in the good years data using the above Shapley value calculation procedure to see if parental favoritism could exist even in the good years. Consider the 21-brood game under the restricted approach where there exists two core chicks as player 1 and 2, and one marginal as player 5. Following the same Shapley value calculation procedure, ordering the chicks in 21-brood into order 1-5-2 means that we firstly give allocation for coalition {1,3} (by giving allocation for chick 1 first from $v(\{1,5\})$ and the rest for chick 5), then lastly give the rest of the grand coalition value $v(\{1,5,2\})$ for chick 2 after being reduced by $v(\{1,5\})$.

Table 3.7 below will list all possible orders as well as the Shapley value for the 21-brood game mentioned earlier in the previous chapter (see Table 2.13 for all the possible coalition values of this 21-brood game). In the end, we also compare the value we got with our observation value for the 21-brood data (see the corresponding average of the survival rate for each chick involves in the 21-brood game from the **A** data). Note that the sum of the observed average of all chicks' survival rate in the corresponding game (see **d8 val** data in table 2.11) is equal to the value of the grand coalition.

Table 3.7: Possible orders for 21-brood game, good years

Possible orders	Player			Total
P	1	2	5	
1-2-5	1,000	0,867	0,598	
1-5-2	1,000	0,603	0,862	
2-1-5	0,867	1,000	0,598	
2-5-1	0,603	1,000	0,862	
5-1-2	0,931	0,603	0,931	
5-2-1	0,603	0,931	0,931	
ϕ	0,834	0,834	0,797	2,465
Observ.	0,859	0,859	0,747	2,465

From Table 3.7, we see that the marginal chick gets a little bit less than its Shapley value solution, while the cores get a little bit more in the observation. It means that according to the Shapley value solution concept and by considering every possible orders of feeding the chicks in the good years 21-brood, we may say that the parents are quite 'favoriting' the core ones.

Now we will leave out the unfeasible orders and consider only the feasible ones. According to our assumption in subsection **2.4.1** that the parents will feed the core chicks firstly before the marginals, we can interchange only the order of the core chicks or the order of the marginals. Coalitions in which any marginal is fed before any core are not feasible. Thus, erasing orders 1-5-2, 2-1-5, 5-1-2, and 5-2-1 from our calculations will give us the table below:

Table 3.8: Feasible orders for 21-brood game, good years

Feasible orders				Total
F	1	2	5	
1-2-5	1,000	0,867	0,598	
2-1-5	0,867	1,000	0,598	
ϕ	0,9335	0,9335	0,598	2,465
Observ.	0,859	0,859	0,747	2,465

We can see in Table 3.8 that if we remove the unfeasible coalition orders using our assumption in subsection **2.2.1**, the result is the other way around. Here the marginal gets much more in reality rather than what it suppose to get based on the Shapley value solution that we calculate.

Now we are questioning about the bad years: would the parental favoritism also appears in the bad years? And if so, would it happen even worse rather than in the good years? We will provide an answer to these questions in the next section.

Interestingly, we can also check whether our Shapley value solutions for the two good years brood games above satisfy the property of a CG-solution as described in subsection **2.1.3**: the one who claims more, will never get less or lose less than the one who claims less. Therefore, we need to check these two conditions on our Shapley value solutions:

1. Whether the one claiming more will always gets more than the one claiming less
2. And whether the one claiming more will always loses more than the one claiming less,

For the case of good years 12-brood game, from Table 3.3 we can see that solely, player 1, 5, and 6 are claiming 1.000, 0.931, and 0.838, respectively. This means, player 1 claims the most, while player 6 claims the least. When considering all possible orders **P**, the Shapley value solutions are 0.960, 0.879, and 0.786, respectively. Since according to these solutions player 1 gets the most while player 6 gets the least, the first property of the CG-solution is satisfied.

However, when we consider the lose (i.e. the difference between the claim and the reward) that every players have, player 1 loses 0.040, while player 5 and 6 equally lose 0.052. The lose of player 1 who claims the most, is in fact smaller than the lose of two other players who claim less. Thus, the second property is unfortunately not satisfied. Therefore, when considering all the possible orders into the calculation, this Shapley value solutions of the good years 12-brood game is not a CG-solution. We can also easily check for the case of removing the unfeasible orders, and may arrive at the same conclusion.

In the case of the good years 21-brood game, we also get the same conclusions when considering only the feasible orders into the Shapley value calculation. However, we get a different result when we consider all the possible orders. Claiming 1.000, 1.000, and 0.931 respectively according to Table 2.13, player 1 and player 2 equally get 0.834, while player 5 gets 0.797 in their Shapley value solutions. Claiming the most, player 1 and 2 lose 0.166, while player 5 loses 0.134. We can easily see that this time, the two properties are satisfied. Thus, we may say that the Shapley value solutions of the good years 21-brood game is a CG-solution when we consider all the possible feeding orders into the calculation.

3.1.2 Calculating Shapley Value of the Bad Years Data

Before we give an example of Shapley value calculation that shows whether the parental favoritism exists even worse in the blackbird family during the bad years, we will provide two tables showing the chicks' average survival rates using the two approaches: the standard and the restricted approach. Note that these two tables are adapted from Table 2.12 in the same way as we adapt the table for the good years in the previous section.

Table 3.9: Survival rates of the players in the existing coalitions (Standard Approach, Bad Years)

		x^S			
XY	S	1 to 4	5	6	7
10	$\{1\},\{2\},\{3\},\{4\}$	1.000	0	0	0
11	$\{1,5\},\{2,5\},\{3,5\},\{4,5\}$	1.000	0.850	0	0
12	$\{1,5,6\},\{2,5,6\},\{3,5,6\},\{4,5,6\}$	0.848	0.839	0.710	0
13	$\{1,5,6,7\},\{2,5,6,7\},\{3,5,6,7\},\{4,5,6,7\}$	1.000	0.931	0.759	0.20
20	$\{i,j\},\forall i,j$:=1 to 4, $i<j$	0.958	0	0	0
21	$\{i,j,5\},\forall i,j$:=1 to 4, $i<j$	0.931	0.686	0	0
22	$\{i,j,5,6\},\forall i,j$:=1 to 4, $i<j$	0.836	0.603	0.233	0
23	$\{i,j,5,6,7\},\forall i,j$:=1 to 4, $i<j$	0.818	0.900	0.300	0.100
30	$\{i,j,k\},\forall i,j,k$:=1 to 4, $i<j<k$	0.815	0	0	0
31	$\{i,j,k,5\},\forall i,j,k$:=1 to 4, $i<j<k$	0.763	0.309	0	0
32	$\{i,j,k,5,6\},\forall i,j,k$:=1 to 4, $i<j<k$	0.917	0.125	0.000	0
40	{1,2,3,4}	0.750	0	0	0
41	{1,2,3,4,5}	0.750	0.333	0	0

Table 3.10: Survival rates of the players in the existing coalitions (Restricted Approach, Bad Years)

		x^S						
XY	S	1	2	3	4	5	6	7
10	{1}	1.000	0	0	0	0	0	0
11	{1,5}	1.000	0	0	0	0.850	0	0
12	{1,5,6}	0.848	0	0	0	0.839	0.710	0
13	{1,5,6,7}	1.000	0	0	0	0.931	0.759	0.207
20	{1,2}	0.958	0.958	0	0	0	0	0
21	{1,2,5}	0.931	0.931	0	0	0.686	0	0
22	{1,2,5,6}	0.836	0.836	0	0	0.603	0.233	0
23	{1,2,5,6,7}	0.818	0.818	0	0	0.900	0.300	0.100
30	{1,2,3}	0.815	0.815	0.815	0	0	0	0
31	{1,2,3,5}	0.763	0.763	0.763	0	0.309	0	0
32	{1,2,3,5,6}	0.917	0.917	0.917	0	0.125	0.000	0
40	{1,2,3,4}	0.750	0.750	0.750	0.750	0	0	0
41	{1,2,3,4,5}	0.750	0.750	0.750	0.750	0.333	0	0

With the same coalition procedure as described in subsection **2.4.2**, we got this coalition table for the 21-brood game during the bad years period:

Table 3.11: 21-brood game, bad years

S	{1}	{2}	{5}	{1,2}	{1,5}	{2,5}	{1,2,5}
v(S)	1.000	1.000	0.850	1.917	1.850	1.850	2.549

Moreover, using similar allocation procedure mentioned in the two examples of the good years data earlier, we build a Shapley value calculation table for the bad years 21-brood data taking into account every possible orders of the players as follows.

Table 3.12: Peasible orders for 21-brood game, bad years

Possible orders	Player			Total
P	1	2	5	
1-2-5	1.000	0.917	0.632	
1-5-2	1.000	0.699	0.850	
2-1-5	0.917	1.000	0.632	
2-5-1	0.699	1.000	0.850	
5-1-2	1.000	0.699	0.850	
5-2-1	0.699	1.000	0.850	
ϕ	0.886	0.886	0.777	2.549
Observ.	0.931	0.931	0.686	2.549

Removing the unfeasible ones, we will get this result:

Table 3.13: Feasible orders for 21-brood game, bad years

Feasible orders				Total
F	1	2	5	
1-2-5	1.000	0.917	0.632	
2-1-5	0.917	1.000	0.632	
ϕ	0.9585	0.9585	0.632	2.549
Observ.	0.931	0.931	0.686	2.549

In the case of taking all possible orders of 21-brood data, the observation results for the average of the core chicks' survival rate are indeed higher, compare to the results from the Shapley value calculations. The other way around happens for the marginals: if we consider all possible orders into account, the marginal's survival rate is 12% higher in calculation rather than in reality. Therefore, we could say that marginal chicks in this case of bad years 21-brood data, seem to be 'stronger' in calculation rather than in the real world. In other words, the parental favoritism exists also in the bad year; and clearly is even worse in the 21-brood game. However, taking the feasible orders show a little bit different result.

We might want to see whether this is always the case for every bad years brood data. However, calculating the same procedures by hand on every XY-brood data during the good and the bad years will be time consuming and not very effective; especially when we arrive on the larger broods like 23-brood game or 32-brood game. Therefore, we will later do computer experiments in order to continue our investigations.

By looking at the three tables above and doing the same checking as in the good years, we found that when we consider every possible orders into the Shapley value calculation for the bad years 21-brood data, the two CG-solution properties are also satisfied.

3.1.3 Calculating the Weights

In the case of this brood data where there exist different weights between the core and the marginal chicks (see subsection **2.4.1**), we need to add different weights w_i and w_j into the fairness equation, in a way that the payoff of one unit is actually worth w_i to player i. Notation w_i denotes the weight of player i, which in this case is the core player in a specific coalition, with i=1 to 4, while notation w_j denotes the weight of the marginal player j, with j:=5 to 7.

Adapting the fairness property mentioned in Formula **2.6** (subsection **2.3.1**) by adding different weights to balance the strength between the two types of the chicks, such that the payoff of one unit is worth w_i to player i, the balanced contributions formula now becomes:

$$\frac{1}{w_i}\left(z_i^S - z_i^{S\setminus\{j\}}\right) = \frac{1}{w_j}\left(z_j^S - z_j^{S\setminus\{i\}}\right) \tag{3.1}$$

In our brood data, setting the weight of the core chick to be equal to one will enable easier calculation for the strength of the marginal chick. We know from the Shapley value calculation before that in most of the cases, there exists a parental favoritism phenomenon. As a results, the marginals may value the food from the parents more than the core. Thus, the corresponding weights for the marginals are predicted to be bigger than the weight of core. Note that the same weight is assumed to apply for all the core chicks, as they may value their food in the same manner. Therefore we may express w_i as w_c, and w_j as w_m, m:=5 to 7, for easier notations.

Substituting our payoff function x^S into the equation, and using the new notations for the core and the marginal chick, we have an adapted balancedness formula for the brood data as follows:

$$\frac{1}{w_c}\left(x_c^S - x_c^{S\setminus\{m\}}\right) = \frac{1}{w_m}\left(x_m^S - x_m^{S\setminus\{c\}}\right) \tag{3.2}$$

Before we go into the computer experiments, we will try to find the corresponding weight for the chicks in each good years XY-brood game by hand. We will see later that calculating the weights for the marginal i chicks by hand does not arrive at a good single solution.

Let us start with the good years 21-brood game. We consider the restricted approach with coalitions $S \in \Omega$, $S = \{\{1\},\{2\},\{1,5\},\{1,2\},\{1,2,5\}\}$. We have a list of payoff vectors for specific coalitions in Table 3.2.

Following the formula above, we get equations for the core and the marginal in the good years 21-brood game as follows.

$$w_c(x_5^S - x_5^{S\setminus c}) = w_5(x_c^S - x_c^{S\setminus 5}) \tag{3.3}$$

$$w_c\left(x_5^{\{c,c,5\}} - x_5^{\{c,5\}}\right) = w_5\left(x_2^{\{c,c,5\}} - x_c^{\{c,c\}}\right) \tag{3.4}$$

Normalizing on the weight of the core w_c=1 and filling in the value of the payoff vectors for the corresponding player in the coalitions $\{c,c,5\}, \{c,5\}$, and $\{c,c\}$

by using the average of survival rates data in Table 3.2, we get this equation for the good years 21-brood data:

$$(0.747 - 0.931) = w_5 \times (0.859 - 0.933) \tag{3.5}$$

Solving the equation above yields solution w_5 =2.865 as the weight of the marginal (i.e. player 5) in 21-brood, while the weights of the core players (i.e. player 1 and 2) are assumed to be always equal to one.

Now, we do the same thing for the 31-brood, to see if we could get the same weight for player 5, w_5, as calculated above.

$$w_c(x_5^S - x_5^{S\backslash c}) = w_5(x_c^S - x_c^{S\backslash 5}) \tag{3.6}$$

$$w_c\left(x_5^{\{c,c,c,5\}} - x_5^{\{c,c,5\}}\right) = w_5\left(x_c^{\{c,c,c,5\}} - x_3^{\{c,c,c\}}\right) \tag{3.7}$$

$$(0.465 - 0.747) = w_5 \times (0.895 - 0.914) \tag{3.8}$$

Solving the equation above yields solution w_5 =14.8421 as the weight of the marginal (i.e. player 5) in the good years 31-brood, with weights of the core players (i.e. player 1, 2, and 3) assumed to be equal to one. This value is indeed not the same as the value that we got before from Equation 3.5.

Below is the resume of the results if we continue calculating all the good years brood data in the same way, by hand:

Table 3.14: Resume of the Marginal Weights

XY-brood	Marginal weight
21-brood	w_5=2.48
22-brood	w_6=-2.59615
23-brood	w_7=-63.25
31-brood	w_5=14.8421
32-brood	w_6=-16.1818
41-brood	w_5=-0.6132

Looking at the results in the table above, we see that we do not arrive at a single weighting solution for the same marginal chick. Not only that the resulting weights for every marginal w_5 and w_6 are different if we compare them for each good years XY-brood data, but also a lot of negative weights popped up. As a result, we need to restrict our Matlab implementation in order to avoid this negative weights.

Perhaps there is a question why we only find the weight of the third marginal w_7 in the good years 23-brood data, but not the weights for the other marginals? Because, if for example we want to find w_5 and w_6 from the 23-brood data, we then need to consider coalition $\{c, c, 6, 7\}$ and coalition $\{c, c, 5, 7\}$ which do not exist/unfeasible since the $(m + 1)$-th marginal can not be hatched before the m-th marginal, m:=1 to 2. In other words, for the cases like this, $S \setminus \{c\}$ exists, but $S \setminus \{m\}$ does not.

3.2 Implementation in Matlab

Jean Derks [18] implements the procedure at Matlab using the three solutions as described in the previous chapter. With respect to the brood data **A** (survival rates) and **I** (importance weights), we are trying to fit the optimal weights for the core and the marginal chicks, in order to get the fitted game with the minimum average of error. We denote datapack **B, G**, and **Rd**, respectively for the bad years, the good years, and the random data.

3.2.1 Without preliminary condition

Knowing that we could not have a single solution for the weight of the marginal chicks in the good years datapack, we begin implementing procedure in Matlab in order to be able to find the best fitted game with smallest error as possible.

Below we show the result of our first experiment implementing only the Shapley value solution on the good years data **G** and the bad years data **B** with 1% average of error as a stopping criteria. Note that here we directly use the chicks' survival rate as our payoff functions, and the search stops when reaching the 1% average of error or when the error $\mathcal{E}(v)$ cannot be reduced anymore. Notation w_i denotes the players, with i:=1 to 4 as the core players, and i:=5 to 7 as the marginal players.

Table 3.15: The resulting weights using Shapley value solution, no restriction

DP	$\mathcal{E}(v)$	w_1	w_2	w_3	w_4	w_5	w_6	w_7
G	0.0899	0.0017	0.0017	0.0017	0.0017	0.3900	0.5654	0.0377
B	0.0722	0.0437	0.0439	0.0437	0.0437	0.4174	0.2191	0.1885

Once we consider to implement a pre-determined direction in which we restrict our solution so that any of the core chick in the data will always have equal weight with the other core chicks. However, we do not do this in our experiments as from the table above we could obviously see that the weights of the marginal chicks are much larger than the cores. Our assumption earlier in the previous chapter holds in this good years data: the core weights appear to be the same as they value the food from their parents in the same manner. There is only a very small difference in one of the core weight regarding the bad years data which can be ignored.

Furthermore, note that the resulting error for both cases are still quite high, while our aim is to find the best fit with the smallest average error. One way that is worth to try in order to see if we can improve the results is, by translating the probability into some kind of utility function. We may see the average of the chicks' survival rate data as a probability, and consider to transform this probability into some kind of utility function that is either concave or convex. We could later see how the results are affected by these different utility translations explained in the next subsection.

3.2.2 With convex and concave translations

As we already mentioned above, now the probability are transformed into utilities p^a, with $a := \{\frac{1}{3}, \frac{1}{2}, 1, 2, 3\}$. If $a < 1$, the transformation follows a concavity increase, while if $a > 1$, it follows the convexity increase. We are not considering utility of p^4 or more, and $p^{1/4}$ or less as the translations seems to be not effective anymore (see Figure 3.2 for the illustration).

By definition, a function is concave if every line segment joining two points on its graph does not lie above the graph at any point [22]. Symmetrically, a function is convex if every line segment joining two points on its graph does not lie below the graph at any point (see pictures taken from [22] below).

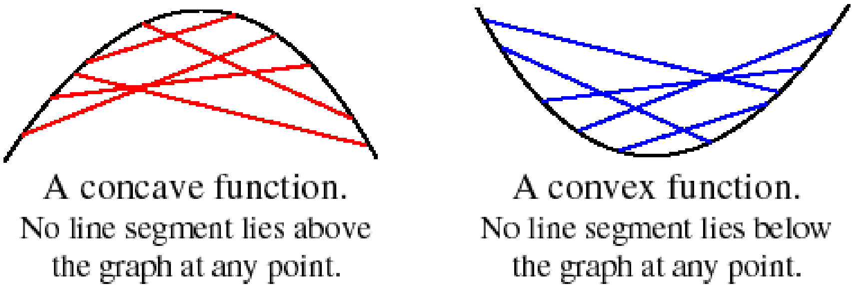

Figure 3.1: Illustration of Concave and Convex Functions

Economists often assume that a firm's production function is concave. The fact that it is concave means that the increase in output generated by a one-unit increase in the input **is smaller when output is large** than when it is small. That is, there are "diminishing returns" to the input.

We predict that the utility function in our brood data is also concave, since the small increase on the food allocation for the marginals, for example, could increase the chick's probability of survive a lot more, and might even save them from dying. In other words, the food is valued more by the marginals rather than the cores. If the parents simply ignore the marginal chicks which have a very small chance to survive by not giving them food at all, in the end there is a much higher probability that the chick will be dead. On the other hand, giving more food to the core chick who already has a high survival rate, does not give a different output as the core already has a great chance of surviving. Thus we think that in this bird brood food allocation problem, the fitting is not linear. To see if our prediction is true, we will test on both translation's directions, as well as the linearity translation, and note on which of these translations we arrive at the best fit.

In Figure 3.2 below, we show the illustration of how this utility transformation works. We also show the utility p^4 and $p^{1/4}$ to illustrate that at this point, the concave transformation is being too close to 0 while the convex transformation is starting to be too close to 1. Thus, we may say that it is not really effective in our case to apply 'too much' transformations by setting p^a with

$a < 1/3$ or $a > 3$.

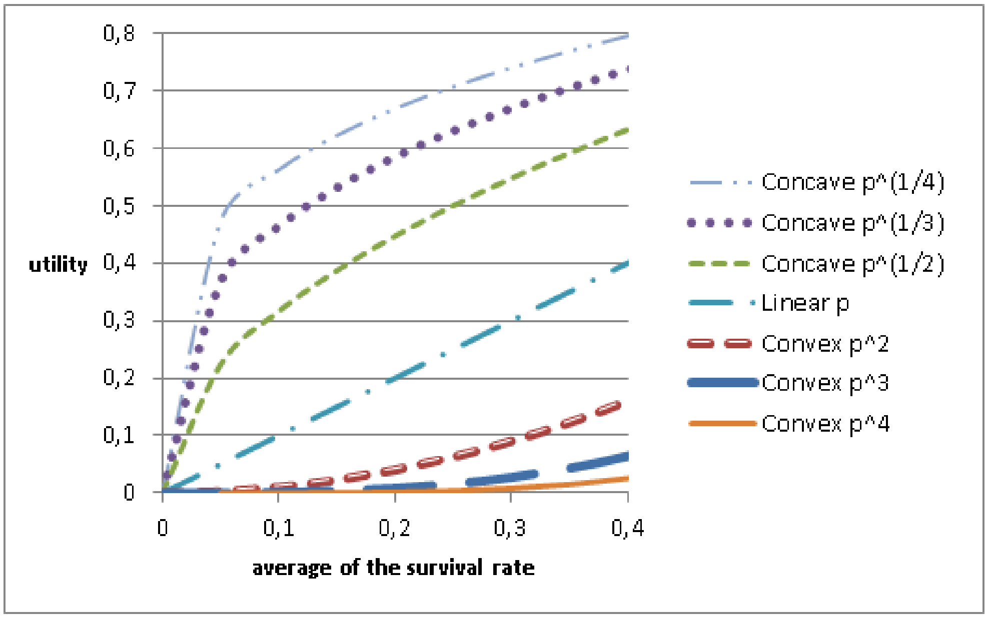

Figure 3.2: Convexity and Concavity Translation of Probability p

Using two different modeling approaches *Appr*, five different utility transformations *UT*, three different solution concepts *Sol*, and three different datapacks *DP*, in total we are doing 90 experiments: 45 experiments for the standard approach **S**, and another 45 for the restricted approach **R**. We will show the results of the experiments for the standard approach using the bad years datapack **B** and the good years datapack **G** with the three solutions described earlier, namely the Shapley Value **Sh**, the Utopia Value **Uv**, and the Prenucleolus **Pr**. Notation w_i, i:=1 to 4, and w_j, j:=5 to 7 denote the weight of the core and the weight of the i-th marginal, consecutively. Notation $\mathcal{E}(v)$ and e_D denote the two error measurements which we calculate by the formula mentioned in equation 2.11 and equation 2.12. Note that we are more interested in the error $\mathcal{E}(v)$ to see how close the brood data we have can fit the solutions of the new game we found.

Standard Approach, Good and Bad years

Firstly we will apply the standard approach as explained in the previous chapter, and conduct thirty experiments for each different cases of the brood datapacks. Considering this standard approach, we assume only the weight of the first core chick to be equal to one, and divide the other weights of the chicks by the weight of the first core to enable easier weights comparison in the experiment results. Note that we may normalize one of the weights to one as the weights refer to utility measurement, and any multiplication or division with a constant factor does not influence the utility differences between the players.

Procedure stops either when we have error lower than 1% average error or when it cannot be reduced anymore (converges to some values). In the first table we will see how the weights of the core players w_i, i:=1 to 4, are distributed in every different cases, then followed by the next table which shows how the weights of the marginal chicks w_j, j:=5 to 7, are compared with the weight of the first core which is normalized to one.

Table 3.16: Experiment results: Standard Approach, core weights

UT	DP	Sol	e_D	$\mathcal{E}(v)$	w_1	w_2	w_3	w_4
1/3	B	Sh	0.0370	0.0328	1	1	2.1350	1
		Uv	0.0277	0.0671	1	1	1	1
		Pr	**0.0256**	**0.0274**	**1**	**1.0905**	**1**	**1.67**
	G	Sh	0.0292	0.0279	1	0.8883	0.8810	0.8810
		Uv	**0.0309**	**0.0162**	**1**	**0.9945**	**0.9945**	**1.0127**
		Pr	0.0211	0.0181	1	0.9897	0.8323	0.9076
1/2	B	Sh	**0.0367**	**0.0336**	**1**	**1**	**1**	**1**
		Uv	0.0426	0.0448	1	1	1	1
		Pr	0.0317	0.0609	1	1.2839	0.7688	1.2839
	G	Sh	0.0367	0.0294	1	1	1.0122	1
		Uv	**0.0405**	**0.0214**	**1**	**0.9610**	**1.0192**	**0.9610**
		Pr	0.0282	0.0295	1	1	0.9945	0.9945
1	B	Sh	**0.0454**	**0.0429**	**1**	**1**	**1**	**1**
		Uv	0.0585	0.0637	1	1	1	1
		Pr	0.0454	0.0584	1	1.1350	1	1
	G	Sh	0.0469	0.0496	1	1	1	1
		Uv	0.0574	0.0540	1	1.0606	1	1
		Pr	**0.0376**	**0.0485**	**1**	**1**	**1.0082**	**1.2015**
2	B	Sh	**0.0565**	**0.0500**	**1**	**0.9945**	**0.9945**	**0.9945**
		Uv	0.1435	0.1720	1	0.9610	0.9610	0.9610
		Pr	0.0591	0.0642	1	1.1400	0.6893	0.8275
	G	Sh	**0.0484**	**0.1189**	**1**	**1**	**1**	**1.0406**
		Uv	0.1108	0.1330	1	1	1	1
		Pr	0.0516	0.1221	1	0.5106	0.5106	0.5106
3	B	Sh	0.0772	0.0553	1	0.9170	0.9543	0.9245
		Uv	0.2732	0.2223	1	2	1	2
		Pr	**0.0693**	**0.0549**	**1**	**1.0606**	**1**	**1.0082**
	G	Sh	**0.0595**	**0.1068**	**1**	**1**	**1**	**1**
		Uv	0.1966	0.1344	1	1	1	1
		Pr	0.0630	0.1482	1	1	1	1

Note that in the two tables below, the corresponding core and marginal weights with minimum error $\mathcal{E}(v)$ for every cases of utility function in the two different brood datapacks are shown in bold. Solutions which give smallest error for each datapacks are also shown in bold.

Table 3.17: Experiment results: Standard Approach, marginal weights

UT	DP	Sol	e_D	$\mathcal{E}(v)$	w_1	w_5	w_6	w_7
1/3	B	Sh	0.0370	0.0328	1	1.6700	2.1350	1.5894
		Uv	0.0277	0.0671	1	2	3	3
		Pr	**0.0256**	**0.0274**	**1**	**1**	**2**	**4.6700**
	G	Sh	0.0292	0.0279	1	1.1288	1.9891	0.9945
		Uv	**0.0309**	**0.0162**	**1**	**0.9829**	**1.0894**	**1.8439**
		Pr	0.0211	0.0181	1	1.4713	1.7621	3.6039
1/2	B	Sh	**0.0367**	**0.0336**	**1**	**2.0606**	**2.1294**	**1.0606**
		Uv	0.0426	0.0448	1	2.0182	1	1
		Pr	0.0317	0.0609	1	1.5376	4.3203	2.1851
	G	Sh	0.0367	0.0294	1	1.0479	1.9220	0.9610
		Uv	**0.0405**	**0.0214**	**1**	**2.0272**	**3.3362**	**1**
		Pr	0.0282	0.0295	1	0.9945	1.9891	1.3018
1	B	Sh	**0.0454**	**0.0429**	**1**	**3**	**3.0905**	**1.0055**
		Uv	0.0585	0.0637	1	2	3.0354	1.0606
		Pr	0.0454	0.0584	1	2.0082	1.0082	2.0182
	G	Sh	0.0469	0.0496	1	1.2621	3.6882	1
		Uv	0.0574	0.0540	1	3.0082	3.0406	3.1350
		Pr	**0.0376**	**0.0485**	**1**	**2.1472**	**2.0406**	**1**
2	B	Sh	**0.0565**	**0.0500**	**1**	**3.0240**	**2.9891**	**3.9782**
		Uv	0.1435	0.1720	1	1.2533	2.0480	2.3587
		Pr	0.0591	0.0642	1	2.1936	0.9610	1.0390
	G	Sh	**0.0484**	**0.1189**	**1**	**3**	**8.0606**	**4**
		Uv	0.1108	0.1330	1	0.7708	3.3239	4.6563
		Pr	0.0516	0.1221	1	2.6782	2	1.1350
3	B	Sh	0.0772	0.0553	1	1.1350	2.1622	1.8050
		Uv	0.2732	0.2223	1	1.9579	4.7700	1.8341
		Pr	**0.0693**	**0.0549**	**1**	**2**	**1**	**3**
	G	Sh	**0.0595**	**0.1068**	**1**	**3**	**5.1350**	**1.2015**
		Uv	0.1966	0.1344	1	2	1	1
		Pr	0.0630	0.1482	1	3.1405	3.7161	1

If we compare the results between each utility translation for both datapacks, we can clearly see that the concave translations (esp. $p^{1/3}$) give the smallest minimum overall error $\mathcal{E}(v)$, thus is supporting our argument that concavity fits best with this brood data.

Taking the smallest minimum overall error $\mathcal{E}(v)$ of the bad years data **B**, which is 0.0274= 2.74% given by the Prenucleolus solution, we arrive at these core weights w_i=(1, 1.0905, 1, 1.67) for each core chicks i=1 to 4, and marginal weights w_j=(1, 2, 4.6700) for each marginals j=5 to 7 respectively, assuming that the weight of the first core w_1 is set to one. For the good years data with the smallest minimum overall error $\mathcal{E}(v)$ of 0.0162=1.62% (given by the Prenucleolus solution), we get these weights w_i=(1, 0.9945, 0.9945, 1.0127) for each core chicks i=1 to 4 and w_j=(0.9829, 1.0894, 1.8439) for each marginals j=5 to 7.

Restricted Approach, Good and Bad years

We now apply the restricted approach **R** into our procedure and conduct another thirty experiments using this approach. Below are the result of the experiments, showing all the corresponding weights of the core and the marginal chicks for each different cases of brood datapacks. Note that the cases with minimum error $\mathcal{E}(v)$ for each datapack are shown in bold.

Table 3.18: Experiment results: Restricted Approach, core weights

UT	DP	Sol	e_D	$\mathcal{E}(v)$	w_1	w_2	w_3	w_4
1/3	B	**Sh**	**0.0303**	**0.0156**	**1**	**1.0409**	**0.9420**	**1.8341**
		Uv	0.0328	0.0164	1	1.3089	1.6700	1.0082
		Pr	0.0222	0.0184	1	1.0905	1	1.8172
	G	**Sh**	**0.0256**	**0.0196**	**1**	**1.1134**	**0.9600**	**3.5387**
		Uv	0.0248	0.0235	1	1.3008	1.0122	2.2867
		Pr	0.0301	0.0223	1	1	2.1194	1.1212
1/2	B	Sh	0.0263	0.0245	1	1	1.2015	4
		Uv	0.0424	0.0278	1	1	2.0678	1
		Pr	0.0368	0.0271	1	1	1.8897	1.2920
	G	Sh	0.0353	0.0317	1	1.4489	2.6298	6.3283
		Uv	0.0371	0.0355	1	1.4713	1.1460	4.2077
		Pr	0.0344	0.0316	1	1	1	4.0488
1	B	Sh	0.0600	0.0387	1	3.0100	2.5211	1.0272
		Uv	0.0531	0.0430	1	1.5000	1.5675	0.5000
		Pr	0.0396	0.0367	1	1	1	3.5811
	G	Sh	0.0582	0.0445	1	0.3851	0.3851	0.5009
		Uv	0.0423	0.0467	1	1.6700	1.0055	3.0612
		Pr	0.0535	0.0443	1	1	1	1.0788
2	B	Sh	0.0486	0.0539	1	2	1	2.0905
		Uv	0.0741	0.0631	1	2	1	2.3512
		Pr	0.0792	0.0622	1	2.0000	1.0272	1.2015
	G	Sh	0.0599	0.0788	1	1.6700	2.6700	2
		Uv	0.0954	0.0872	1	1.3083	0.5222	0.5654
		Pr	0.0607	0.0639	1	1.2015	2	4.5023
3	B	Sh	0.0360	0.0971	1	1.4489	1	1.2676
		Uv	0.0886	0.0994	1	1.1015	0.5269	1.5365
		Pr	0.1406	0.1056	1	1.5726	0.5237	2.0459
	G	Sh	0.1380	0.1147	1	1	1	1.0177
		Uv	0.1143	0.1449	1	0.3620	0.3190	0.3449
		Pr	0.0746	0.1548	1	0.5379	0.4933	0.4960

Table 3.19: Experiment results: Restricted Approach, marginal weights

UT	DP	Sol	e_D	$\mathcal{E}(v)$	w_1	w_5	w_6	w_7
1/3	B	**Sh**	**0.0303**	**0.0156**	**1**	**1**	**2.3400**	**1.6700**
		Uv	0.0328	0.0164	1	1.2436	1.4804	1.2264
		Pr	0.0222	0.0184	1	1.1350	1.0606	2.1476
	G	**Sh**	**0.0256**	**0.0196**	**1**	**0.9170**	**0.9170**	**1**
		Uv	0.0248	0.0235	1	1.3008	1,6700	1
		Pr	0.0301	0.0223	1	1	1	2.3008
1/2	B	Sh	0.0263	0.0245	1	1.0905	2.2255	1.2255
		Uv	0.0424	0.0278	1	1	2.1350	1
		Pr	0.0368	0.0271	1	1	1.0678	1.5501
	G	Sh	0.0353	0.0317	1	1	2	1
		Uv	0.0371	0.0355	1	1	4.0834	1.2015
		Pr	0.0344	0.0316	1	1	2.2015	2.0082
1	B	Sh	0.0600	0.0387	1	1.4713	4.1891	1.8553
		Uv	0.0531	0.0430	1	0.7853	0.3851	1.9253
		Pr	0.0396	0.0367	1	2.4489	3.3467	2.1795
	G	Sh	0.0582	0.0445	1	1.0272	2	1.0272
		Uv	0.0423	0.0467	1	1	1.8456	1
		Pr	0.0535	0.0443	1	1.0082	2	1.3089
2	B	Sh	0.0486	0.0539	1	2.6868	2.4724	1.1504
		Uv	0.0741	0.0631	1	0.7635	0.5298	1.0072
		Pr	0.0792	0.0622	1	2.4489	2	3.6015
	G	Sh	0.0599	0.0788	1	2.0905	3.5448	2
		Uv	0.0954	0.0872	1	1.3157	1.0171	0.4645
		Pr	0.0607	0.0639	1	1.6015	3.0905	4.5927
3	B	Sh	0.0360	0.0971	1	3.6800	3	2.3400
		Uv	0.0886	0.0994	1	0.9859	0.7527	0.9859
		Pr	0.1406	0.1056	1	0.3190	0.7339	0.5539
	G	Sh	0.1380	0.1147	1	1.4544	3.7110	2
		Uv	0.1143	0.1449	1	1.0905	2.0164	2.3159
		Pr	0.0746	0.1548	1	0.4933	2.5311	1.1648

We can clearly see that the concave translation $p^{1/3}$ here also gives the minimum overall error $\mathcal{E}(v)$ for both datapacks, just like the result of our previous standard approach. However, unlike the previous approach, the overall minimum error of the bad years datapack and the good years datapack in this restricted approach are both given by the Shapley value solution. We will provide more analysis regarding these results in the next chapter.

Standard and Restricted Approach, Random data

To see how the experiment goes on random data, we use the random datapack where the survival rate of the chicks are generated by the **'rand'** function in

Matlab and are uniformly distributed in the interval [0,1]. The resulting errors for both approaches are shown in tables below:

Table 3.20: Experiment results: Standard Approach, random data

UT	Sol	$\mathcal{E}(v)$	e_D
1/3	Sh	0.0920	0.0862
	Uv	0.1062	0.1309
	Pr	0.0762	0.0850
1/2	**Sh**	**0.0242**	**0.06261**
	Uv	0.0946	0.0934
	Pr	**0.0215**	**0.0686**
1	Sh	0.2101	0.1100
	Uv	0.2840	0.2337
	Pr	0.0725	0.0802
2	Sh	0.2555	0.2659
	Uv	0.7029	0.3539
	Pr	0.5768	0.3054
3	Sh	1.1263	0.3443
	Uv	13.1960	0.1501
	Pr	20.0752	0.3173

Table 3.21: Experiment results: Restricted Approach, random data

UT	Sol	$\mathcal{E}(v)$	e_D
1/3	Sh	0.0549	0.0690
	Uv	0.1178	0.1003
	Pr	0.0726	0.0677
1/2	Sh	0.0560	0.0602
	Uv	0.1033	0.0849
	Pr	0.0718	0.0344
1	Sh	0.3683	0.1152
	Uv	0.6478	0.2361
	Pr	0.1618	0.1511
2	Sh	4.8517	0.2181
	Uv	15.9842	0.3801
	Pr	3.8706	0.2544
3	Sh	0.6727	0.2697
	Uv	0.4723	0.4718
	Pr	0.6913	0.2260

As shown in bold, we can easily see from Table 3.20 and 3.21 above that the overall minimum errors on the random data hit below 5% in only two cases of the standard approach, and never on the restricted approach. As a result, the weights of the chicks on the random data become none of our interest since the two errors $\mathcal{E}(v)$ and e_D are quite high in most of the cases.

Looking closely at the error $\mathcal{E}(v)$, notice that a different tendency appears in these random dataset solutions compare to the brood datasets. The minimum $\mathcal{E}(v)$ error for the standard approach is achieved by $p^{1/2}$ translation using the Prenucleolus solution, with error of 0.0215=2.15%, followed by the Shapley value with error of 0.0242=2.42%. While the Utopia value solution seems to have larger error with minimum of 0.0946=9.46%.

For the restricted approach, the concave utility function of $p^{1/3}$ gives the smallest error $\mathcal{E}(v)$ using the Shapley value solution with error of 0.0549=5.49%. However, this error is probably still quite large for the result to be considered as a best fit.

In general, unlike the brood datapacks which has smallest minimum $\mathcal{E}(v)$ error under the $p^{1/3}$ translation, both approaches on the random data give quite a high error under this $p^{1/3}$ translation, ranging from 5.5% to 12%.

Interestingly, we can also highlight on the largest minimum $\mathcal{E}(v)$ error for the standard approach which is given by the Prenucleolus solution under the convex p^3 translation, with 20.0752=2007.52% of overall error. While for the restricted approach, the largest minimum $\mathcal{E}(v)$ error is achieved by the Utopia solution with a convex p^2 translation, resulting 15.9842=1598% of error. Both are naturally unacceptable. Therefore, we cannot say that the random data fits any solution concepts mentioned in this thesis.

Note that in the first experiment with no restriction, we want to see whether we can achieve smaller error below 1%. However, this never occurs in any of the 90 experiments above; none of them stops at the 1% average of error.

3.3 Validation

When finding the best weight system using the standard approach, we validate the resulting weights of the core chicks by taking a general direction instead of a predetermined one. Doing this will allow us to see if by any chance, the weights of the core chicks will be the same or at least close to each other. Indeed, we found from Table 3.16 and Table 3.18 that in almost all cases where the error is small, the weights of the other core chicks appear to be the same as the weight of the first core chick, which we set to be equal to one. While in most cases where the error is quite small, the weights of the core chicks are close to one; i.e. almost the same with the weight of the first core.

Once, we had an experience with some strange results. The resulting weights of the prenucleolus and the utopia solution concepts are not 'going in the same direction' as the results from our Shapley value solution: in any cases of the brood data, almost all weights of the marginal broods are smaller than the weight of the core. Logically, the weights of the marginals will usually be larger than the core as a results of parental favoritism where food is valued more by the marginals rather than the core (see subsection **3.2.2** for the detailed explanation).

In order to validate, we run our 'old' codes which is made only for fitting the game using the Shapley value solution concept (see [18] for different versions of these fitting procedures), indeed we got the same tendency with our Shapley value results: the weights of the marginals are almost always larger than the core. Therefore, we check and correct our codes for the prenucleolus and utopia solutions by adapting the same procedure with the Shapley value. Now our results for all the approaches seem to be more convincing as the marginal weights have larger weights in most of the cases.

Chapter 4

Further Analysis

This chapter considers the analysis of the results from computational and biological point of view, to see which properties seem to play a major role in the brood data and which known solution concept fits best with the brood data.

4.1 Notations

To be able to read the tables mentioned in the analysis of this chapter, here we provide again the same notations as we use in the previous chapter. *Appr* denotes the modeling approach we are using, **S** for the Standard approach and **R** for the Restricted approach. *DP* denotes which datapack we are taking into account, **B** for the Bad years data, **G** for the Good years data, and **Rd** for the Random data. Notation *Sol* denotes what kind of solution concept we choose, **Sh** for the Shapley value, **Uv** for the Utopia value, and **Pr** for the Prenucleolus. Notation *UT* denotes which utility translation we are considering, either $p^{1/3}$, $p^{1/2}$, p^1, p^2, or p^3. Notation $\mathcal{E}(v)$ denotes the minimum overall error, which express how well the data fits the new fitted game, while e_D denotes the error with respect to the data, expressing how well the game solutions fit the data.

4.2 Findings and Reasoning

We divide the findings we got from the experiments into seven different types of analysis, and explain the details of the analysis in seven different sections below. We will also give some reasonable arguments behind the results we found in the experiments.

4.2.1 Smallest Error vs Utility Translation

As we have known from the results of our experiments in the previous chapter, smallest errors are reached by the concave $p^{1/3}$ translation in all cases of the brood data solutions with two different modeling approaches. Thus, we may already say that the brood survival rate data in our case is better translated with concavity of $p^{1/3}$. For a clear comparison on the minimum overall error for every different cases of utility translations, see the table below. Note that the minimum overall error for two different approaches and datapacks are shown in bold.

Table 4.1: Smallest Error vs Utility Translation

			$\mathcal{E}(v)$				
Appr	DP	Sol	$p^{1/3}$	$p^{1/2}$	p^1	p^2	p^3
S	B	Sh	0.0328	0.0336	0.0429	0.0500	0.0553
		Uv	0.0671	0.0448	0.0637	0.1720	0.2223
		Pr	**0.0274**	0.0609	0.0584	0.0642	0.0549
	G	Sh	0.0279	0.0294	0.0496	0.1189	0.1068
		Uv	**0.0162**	0.0214	0.0540	0.1330	0.1344
		Pr	0.0181	0.0295	0.0485	0.1221	0.1482
R	B	Sh	**0.0156**	0.0245	0.0387	0.0539	0.0971
		Uv	0.0164	0.0278	0.0430	0.0631	0.0994
		Pr	0.0184	0.0271	0.0367	0.0622	0.1056
	G	Sh	**0.0196**	0.0317	0.0445	0.0788	0.1147
		Uv	0.0235	0.0355	0.0467	0.0872	0.1449
		Pr	0.0223	0.0316	0.0443	0.0639	0.1548

We can see from the table that the convex translation p^2 hit 5% overall error only once with the Shapley value solution concept, and never with the other solution concepts. While in the convex p^3 translation, the minimum overall error is always above 5% for any solution concepts.

Taking a close look into the results using the Shapley value solution concept in the table above, we may notice that the more concave the utility translation p^a $(a = \{1/3, 1/2, 1, 2, 3\})$ is, the smaller the error. For the Utopia value and the Prenucleolus solutions, sometimes the concave $p^{1/2}$ translation gives smaller error than the concave $p^{1/3}$ translation. However, concave translations in general always give smaller error than the convex ones.

Why concave translation fits best? As we have argued before in the previous chapter, if the food allocation for the marginal chick is increased a little bit more, for example from 0.3 to 0.4 gram, the chick's probability of survive will increase a lot more, and it might even save them from dying. While if the core chick's survival rate is already high, there is almost no difference if it is being fed a little bit more; in any chance, most probably it will survive. In other words, the food is considered to be more valuable for the marginals rather than for the core. This may explains why the concave translation fits best than the convex one.

4.2.2 Smallest Error vs Solution

The smallest overall error for the good years and the bad years data in the standard approach are given by the Utopia value and the Prenucleolus solutions, respectively. While for the restricted approach, the smallest overall error for both datapacks are given by the Shapley value solution. See the table below for the five first smallest errors of the solutions for the two approaches.

Table 4.2: Smallest Error vs Solution

Appr	Rank	$\mathcal{E}(v)$	e_D	UT	DP	Sol
S	1	0.0162	0.0309	$p^{1/3}$	G	Uv
	2	0.0181	0.0211	$p^{1/3}$	G	Pr
	3	0.0214	0.0405	$p^{1/2}$	G	Uv
	4	0.0274	0.0256	$p^{1/3}$	B	Pr
	5	0.0279	0.0292	$p^{1/3}$	G	Sh
R	1	0.0156	0.0303	$p^{1/3}$	G	Sh
	2	0.0164	0.0328	$p^{1/3}$	B	Sh
	3	0.0184	0.0222	$p^{1/3}$	G	Pr
	4	0.0196	0.0256	$p^{1/3}$	B	Uv
	5	0.0223	0.0301	$p^{1/3}$	B	Pr

We can see from the table above that the Utopia value solution concept gives smallest overall error compare to the other two solution concepts if we are using the standard approach. It supports our prediction that the Utopia value solution gives a better fit in the standard approach as it has more freedom and not as refine as the Shapley value or the Prenucleolus. It also uses less game data.

On the other hand, Shapley value gives smallest overall error in the restricted approach. As we have already known from subsection **2.2.1**, all the core chicks are considered to be equal with each other. Not only they are hatched on the same day thus is 'stronger' compare to the marginals, but also they are assumed to always being fed at the first place with almost the same amount of food by the parents. Perhaps, this 'special' structure of the brood datasets that are implicitly translated into the restricted approach by setting an equal weight for all the core chicks and assuming i number of core to always fill the first i position in the coalition (i=1 to 4), are recognized by the Shapley value solution concept; thus is facilitating its fitting procedure to give a better fit for the restricted model compare to the other two solution concepts.

4.2.3 Smallest Error and The Weight System

According to the solutions which has smallest minimum error $\mathcal{E}(v)$ for both good and bad year data in two respected model systems within the concave translation $p^{1/3}$, we get these corresponding weights as follows. Notice that w_1 column is left out as the weight of the first core chick is normalized to 1 for easier comparison.

Table 4.3: Smallest Error and The Weight System

Appr	DP	$\mathcal{E}(v)$	e_D	w_2	w_3	w_4	w_5	w_6	w_7
S	B	0.0274	0.0256	1.09	1	1.67	1.67	2.14	1.59
	G	0.0162	0.0309	0.99	0.99	1.01	1.13	1.99	0.99
R	B	0.0164	0.0328	1.31	1.67	1.01	1	2.34	1.67
	G	0.0156	0.0303	1.04	0.94	1.83	0.92	0.92	1

What we want to see is whether the weights of all core chicks will be the same (i.e. close to each other), at least in the standard approach where we allow anonimity between the core chicks. By dividing all the weights of the other chicks using the weight of the first core chick, we see that almost all the other core weights are very close to 1 in the standard approach with the smallest error. Interestingly, this is also the case for the restricted approach. Even though the first four players do not take similar position in the game, we see that similar assumption seems to hold for the restricted case, especially when the error is small: the weights of the other core chicks are most of the time close enough to 1.

However, we may see some unexpected values appearing on the weight of the core and the marginal broods: some weights of the core chicks appear to be quite larger than one, and some weights of the marginal chicks are a little bit smaller than one. This is due to the fact that our search procedure is not 'perfect'. During the fitting of the weight, sometimes we end up in a local optima and the procedure does not allow us to search further to find the best fit. Given the search procedure, we cannot do better. However, most of the time when error is below 5%, this is usually not the case.

4.2.4 The Brood Data and The Weight System

Now we compare the weight system in different brood datapacks for the two approaches. We want to check whether the marginals have larger weights in the bad years compare to the good years, at least for the case where the average error is minimum. Note that the largest weights of the marginals are shown in bold.

Table 4.4: The Brood Data and The Weight System

Appr	DP	$\mathcal{E}(v)$	e_D	w_5	w_6	w_7
S	B	0.0274	0.0256	**1.67**	**2.14**	**1.59**
	G	0.0162	0.0309	1.13	1.99	0.99
R	B	0.0164	0.0328	**1**	**2.34**	**1.67**
	G	0.0156	0.0303	0.92	0.92	1

We can easily see that in the case where the overall average error $\mathcal{E}(v)$ is minimum, the marginal chicks have larger weights in the bad years rather than in the good years. These results support our prediction that parental favoritism exists even worse during the bad years period.

4.2.5 Random Data vs The Brood Data

For both approaches on the random data using the concave $p^{1/3}$ translation, we arrive at error larger than 5%. Below we will show the minimum overall error $\mathcal{E}(v)$ of the random data for each solution concept and compare them with the minimum overall error of the brood data during the good and the bad years. Note that the largest error $\mathcal{E}(v)$ and e_D for every different solution concepts considered by the two different approaches in each datapacks are shown in bold.

Table 4.5: Random Data vs The Brood Data

DP	Appr	Sol	$\mathcal{E}(v)$	e_D
Rd	S	Sh	**0.0920**	**0.0862**
		Uv	**0.1062**	**0.1309**
		Pr	**0.0762**	**0.0850**
	R	Sh	**0.0549**	**0.0690**
		Uv	**0.1178**	**0.1033**
		Pr	**0.0726**	**0.0677**
G	S	Sh	0.0279	0.0292
		Uv	0.0162	0.0309
		Pr	0.0181	0.0211
	R	Sh	0.0303	0.0156
		Uv	0.0263	0.0245
		Pr	0.0222	0.0184
B	S	Sh	0.0328	0.0370
		Uv	0.0671	0.0277
		Pr	0.0274	0.0256
	R	Sh	0.0164	0.0328
		Uv	0.0196	0.0256
		Pr	0.0223	0.0301

As shown in the table above, the error $\mathcal{E}(v)$ and e_D for the random data are always larger compare to the brood data in both good and bad years, no matter which approach or which solution we are taking into account. Therefore we may say that the random data cannot fit into any of the three solution concepts we studied in this thesis.

4.2.6 Largest Error: Worse Case

As we have mentioned earlier in the previous section, the largest minimum overall error is achieved by the standard model for the random data using the Prenucleolus solution concept with high level of convexity p^3, resulting in an unacceptable 2007,52% overall error. It is possibly because of these four reasons:

1. There are more coalitions to be considered in the standard approach rather than in the restricted approach, thus the fitting procedure may take much longer time to find the best fit and is then more likely to make mistake by arriving at the local optima.

2. It is hard to fit the random data since there is no specific structure, unlike the brood datasets which have some 'special' properties that can be captured by some of the solution concepts.

3. The Prenucleolus solution concept itself contains a lot of sophisticated tuning and balancing, unlike the Utopia value or the Shapley value solutions which have the additivity property that enables a fast fitting.

4. The utility translation is convex instead of concave; in which we already prove that concave translations $p^a, a = \{1/3, 1/2\}$, usually give smaller minimum overall error $\mathcal{E}(v)$ rather than the convex translation $p^a, a = \{2, 3\}$ or the linear case without translations.

4.2.7 Time Consumption

As an addition to the analysis, below we will show the largest time consumption for every solution concept in each brood datapack which we have not mention in the previous chapter. Note that the two largest time consumptions (in seconds), each corresponds to the standard and the restricted approach, are shown in bold.

Table 4.6: Time Consumptions

Appr	DP	Sol	UT	time
S	B	Sh	3	596,6758815
		Uv	1/2	1258,745863
		Pr	1	5482,497638
	G	Sh	2	1433,950145
		Uv	1	5436,586818
		Pr	**1/3**	**6133,127812**
R	B	Sh	1/2	6294,65302
		Uv	1/3	1433,452379
		Pr	2	4525,036143
	G	Sh	1	109,2940425
		Uv	2	4140,893783
		Pr	**1/2**	**6470,758277**

We see that the largest time consumptions in both approaches happen when we consider the Prenucleolus solution, especially under the concave translations.

As an insight on the time consumption of the experiments, fitting using the Prenucleolus solution concept is much slower compare to the other solution concepts perhaps because the Prenucleolus does not have special properties like linearity that can be explored in the implementation of the fitting procedure.

Chapter 5

Final Conclusion

In this last chapter we provide conclusions and future works of our research.

5.1 Conclusions

As Forbes (2011) said in his paper [16], survival to leave the nest is a useful proxy to measure the evolutionary fitness. It shows a range of variation, from the virtually guaranteed survival of core offspring in small broods, to the near certain death of marginal offspring in large broods. Knowing the importance of this survival rate for the blackbird chicks, we conclude our research as follows:

- The modeling and calculation choices we made using the Shapley value solution concept have proved that parental favoritism does exist in most cases of the brood data, especially during the bad years period.
- In general, property such as linearity in the Shapley value solution allows a fast fitting in the brood data. The restricted structure of the brood datasets also enables the Shapley value solution concept to give a reasonable fit within a reasonable time.
- Prenucleolus solution results are shown in various kind of experiments, perhaps because of the fact that it is a high profile solution where there are quite a lot of balancing in the procedure, i.e. fine tuning is sophisticated.
- If we want a simple, 'quick and dirty' solution which also enable fast fitting with quite feasible solution, then Utopia value solution concept can be considered as a choice.
- Whatever model of the brood data we are taking, the results of the experiments have some tendencies:
    1. The weights of the marginals are usually larger than the cores, since the food is valued more by the marginals as a result of parental favoritism.

2. The concavity translations fit best as our brood data is not linear. Small increase on the food allocation for the marginals could increase the chick's probability of survive a lot more, while giving more food to the core chick who already has a high survival rate does not give a different output as the core already has a great chance of surviving.

- The worst case of error is achieved by the standard model for the random data using the Prenucleolus solution concept with high level of convexity p^3. Thus, the more complex the model, the harder it fits.
- To summarize the results of the experiments, we can say that the more concave the utility translation, the smaller the error, the better the game fits the solutions, the more we can trust the resulting weight system.

Finally we conclude that we have been able to implement the Shapley value solution concept to tackle the bird brood food allocation problem, and compare its result with the other known solution concepts namely the Utopia value and the Prenucleolus.

We also successfully translate the biological problem of blackbird food allocation into a cooperative game approach using various techniques known in literature.

5.2 Future Works

Given a vast amount of brood data as well as a vast amount of results from the experiments, there are still a lot to be investigated further in this research. One can really discriminate among the approaches, the solutions, the utility functions, and the datapacks that are used, and analyze the results for each experiment.

To get a better understanding, one might also change the solution concept and examine the effects in the current situation. To some extent, we adapted the anonimity property of the Shapley value by applying weights. What will happen if we furthermore play around with the dummy player property, or when we allow for restricted cooperation settings, as are seen in the Utopia value. In all these cases we may arrive at more weight in the importance of the game solution properties in the interpretation of the data.

The fitting procedure may also be tuned in a different way. The applied settings were tuned only briefly and only in the Shapley value application. A more sophisticated calibration might enable a fitting with smaller erroors, and thus with more reliable conclusions.

Bibliography

[1] S. Forbes, *A Natural History of Families*, Princeton University Press, 9 May 2005

[2] S. Forbes, ”Sibling Symbiosis in Nestling Birds”, *Auk 124*, pp. 1-10, 2007

[3] A. Kacelnik, et al, ”Food allocation among nestling starlings: sibling competition and the scope of parental choice”, in *The Royal Society B*, Biological Sciences, (London, U.K.), vol. 259, no. 1356, pp. 259-263, 22 March 1995

[4] V.F. Hendricks and P.G. Hansen, ”Game Theory: 5 Questions”, *Automatic Press/VIP*, April 2007 http://www.gametheorists.com/Interviews/young.html

[5] C-J. Haake, A. Kashiwada, and F.E. Su, ”The Shapley value of phylogenetic trees”, working papers, Bielefeld University, Institute of Mathematical Economics, 2005, http://EconPapers.repec.org/RePEc:bie:wpaper:363

[6] E. Binenbaum and P.G. Pardey, ”Collective Action in Plant Breeding”, in Proc. *2005 annual meeting, Providence, RI, American Agricultural Economics Association (New Name 2008: Agricultural and Applied Economics Association)*, no. 19530, 24-27 July 2005, http://EconPapers.repec.org/RePEc:ags:aaea05:19530

[7] P. Pintassilgo, ”The New Member Problem in the Cooperative Management of the Northern Atlantic Bluefin Tuna”, FEUNL Working Paper Series, Universidade Nova de Lisboa, Faculdade de Economia, 2000, http://EconPapers.repec.org/RePEc:unl:unlfep:wp373

[8] A. Torkaman, N.M. Charkari, and M. Aghaeipour, ”An approach for leukemia classification based on cooperative game theory”, *Anal Cell Pathol*, (Amsterdam, The Netherlands), vol. 34, no. 5, pp. 235-46, 2011

[9] F. Thuijsman, ”To play and to share; a brief introduction to game theory”, lecture notes, Maastricht University, Department of Knowledge Engineering, 2011

[10] R.J. Aumann and M. Maschler, "Game Theoretic Analysis of a Bankruptcy Problem from the Talmud", *Journal of Economic Theory*, no. 36, pp. 195-213, 1985

[11] A. Kar, M. Mitra, and S. Mutuswami, "On the coincidence of the prenucleolus and the Shapley value", *Mathematical Social Sciences*, Elsevier, vol. 57, no. 1, pp. 16-25, January 2009

[12] J. Derks, "A cooperative game approach to the analysis of allocation sequences", discussion notes, Maastricht University, Department of Knowledge Engineering, 26 September 2008

[13] H. Wiese, "Applied cooperative game theory: Dividends", lecture notes, University of Leipzig, April 2010

[14] J.J.M. Derks and H.H. Haller, "Weighted Nucleoli", *International Journal of Game Theory*, no. 28, pp. 173-188, 1999

[15] J. Oswald, J. Derks, and H. Peters, "Prenucleolus and Nucleolus of a Cooperative Game: Characterizations by Tight Coalitions", in *Proc. 3rd International Conference on Approximation and Optimization in the Caribbean*, (Puebla, Mexico), 8-13 October 1995

[16] S. Forbes, "Parental preference for investment risk incites family strife", *Journal of Bioeconomics*, Springer, (Netherlands), vol. 14(2), pp. 115-128, 1 July 2012, http://dx.doi.org/10.1007/s10818-011-9122-0

[17] T.S. Ferguson, "Game Theory", electronic text, UCLA, Mathematics Department, 2008

[18] J. Derks, "Cooperative Game Fitting Procedures", Matlab toolbox package, April 2012

[19] A.Ph. Rosenbusch, "Fairness Considerations in Cooperative Games", lecture notes, TU Darmstadt, Department of Mathematics, April 2011, http://tuprints.ulb.tu-darmstadt.de/2573/

[20] H. Keiding, "Solution concepts for TU games: The τ-value", lecture notes, University of Copenhagen, Department of Economics, pp. 1-4, http://www.econ.ku.dk/keiding/underv/GameTheory/tijsrevEng.pdf

[21] S. Brnzei, "State of the Art: Solution Concepts for Coalitional Games", *BRAIN (Broad Research in Artificial Intelligence and Neuroscience)*, vol. 1, no. 2, University of Waterloo, David R. Cheriton School of Computer Science, 2010

[22] M.J. Osborne, "Mathematical methods for economic theory: a tutorial", electronic tutorial, University of Toronto, Department of Economics, 28 November 2011

[23] L.S. Shapley, A Value for n-Person Games, "In Contributions to the Theory of Games", *Annals of Mathematics Studies*, vol. II, no. 28, pp. 307-317, H.W. Kuhn and A.W. Tucker, eds., Princeton University Press, 1953

[24] D. Schmeidler, "The nucleolus of a characteristic function form game", *Defense Technical Information Center*, pp. 1-13, Hebrew University Jerusalem (Israel), Department of Mathematics, 1966

[25] J.J.M. Derks and H.M. Peters, "Orderings, Excess Functions, and the Nucleolus", *Mathematical Social Sciences*, no. 36, pp. 175-182, 1998

[26] R.v.d. Brink, "Balanced consistency and balanced cost reduction for sequencing problems", *Social Choice and Welfare*, Springer-Verlag, vol. 38, no. 3, pp. 519-529, March 2012, http://dx.doi.org/10.1007/s00355-011-0533-6

[27] D. Monderer, D. Samet, and L.S. Shapley, "Weighted Values and the Core", *International Journal of Game Theory*, vol. 21, pp. 27-39, 1992