Master Thesis

Decision making based on sequences

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Preface

This thesis is the result of the research performed for the Masters degree in Operations Research at Maastricht University. The research was carried out under supervision of Dr. F. Thuijsman and Dr. J. Derks.

Special thanks go to all people who helped me complete this thesis. In particular my two supervisors, who made a lot of effort in giving my thesis the right direction. I thank my family and all my friends who studied together with me, for their ideas, questions, suggestions and support.

Finally, I hope the reader enjoys this thesis.

Abstract

This thesis focuses on optimal stopping and secretary problems. These are problems where a decision maker is confronted with a sequence of applicants for a secretarial position, who are observed one by one, while knowing the length of the sequence. The goal is to pick out the best applicant, but immediately after observing an applicant it has to be decided whether to accept or reject her. Only one applicant can be accepted and applicants once rejected can not be recalled. The question is: What strategy maximizes the probability of accepting the best applicant? While this problem can be solved analytically, for several variants of the problem no solutions are known. In this thesis several of these problems are investigated and a simulation environment is discussed that is developed to determine optimal strategies for those variants for which no analytical solution is known.
The variants that are examined differ from the above formulation in several ways. Instead of being interested in getting the very best only, it may be interesting to get one of the best, or simply to maximize the cardinal value of the accepted secretary. In terms of the rules, it is also examined what happens if the decision maker is allowed to accept or reject candidates with some delay. As another variant the problem of approximating the mean of the sequence, in as few observations possible, is analyzed by including costs for each separate observation and by defining a payoff in terms of the distance from the actual mean of the sequence.
The stopping problem variants with known outcomes are validated with the simulation environment and outcomes are observed for which no analytic result is known. The influence of different payoff functions, strategies and distributions of the sequences are investigated. A lot of new variants of stopping problems can be examined with the concepts given in this thesis, and the created simulation environment.

Keywords: optimal stopping, secretary problems, strategies.

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Chapter 1: **Introduction**

In this chapter the subject of the present study will be introduced. In the first section the reader will become acquainted with the theory of making decisions based on sequences. These decision problems are known as stopping problems. The second section describes available strategies to find important values of a sequence for several purposes, as they are used in recent years. Also the problem faced in this thesis is presented here. The third section treats the research goals, after which in the last section an outline of this paper is given.

1.1 General introduction

The prediction of outcomes of surveys, games, and elections is a hot topic nowadays. People want to know the outcome of elections or games before all data is collected and verified. Regard for example the election of the president of the United States of America. As soon as a certain percentage of the votes is known, people want to know who is leading the elections and the chance that this leader is going to be the next president of the USA [10]. Another example comes from the business world. At Statistics Netherlands[1] monthly growth rates of several enterprise groups have to be published between the 23th and the 27th day of the next month. These growth rates are based on VAT-information. According to the law, this VAT-information is to be declared at the tax office between the 1st and 30th day of the next month. Unfortunately, most enterprises declare their VAT just before the deadline. This means that the growth rate can not be based on full VAT-information [18]. The earlier the growth rate is known by the enterprise group, the more satisfied they are. But of course they are only satisfied if the growth rate information based on incomplete data is similar to the growth rate information based on complete data. What strategies can be used to predict the growth rates only on available VAT-information?
Both examples may be seen as decision problems based on sequences. In both cases the total number of data is known in advance, but a decision or prediction has to be made before all data are collected. How reliable is the information given, based on these data, when observing new data is stopped early? What are the costs for observing more data than necessary before making a decision? The theory that deals with these problems is known as optimal stopping [8, 17]. Optimal stopping in itself is a wide topic in mathematics. In the research described in this thesis, a program is developed in which many variants on the famous secretary problem, outlined in section 1.2, can be simulated in order to maximize the probability of finding the maximum of a sequence. The program also contains a method to maximize the probability to give a good estimation of the mean of a sequence, while not all data of the sequence are known.

1.2 Stopping problems

While analyzing sequences, interesting and important values are the maximum, the minimum and the mean. The reason is shown in the following example, which is related to the game of googol described in a column by Gardner in 1960 [9]. Suppose a situation where a boy may grab in a grab bag which contains a known number of cheques which all have a different value. He wants to grab the highest valued cheque out of the grab bag. After grabbing a cheque,

[1] In Dutch: Centraal Bureau voor de Statistiek (CBS)

he has to decide to keep the cheque and get its value, or to reject that cheque (and get no money), and grab a next cheque. The goal of the boy is to get the cheque with the highest value. He does not know anything in advance about the distribution of the values or the actual values. To grab the maximum valued cheque, there are many possible strategies. The strategy of grabbing the highest valued cheque while taking into account the described rules, is similar to the secretary problem stated by Ferguson [7]. In literature this is also known as the marriage problem [13], the house hunting problem [6] and several other synonyms. The secretary problem appeared in the late 1950's and early 1960's and made its way around the mathematical community. In its simplest, classical form the secretary problem is stated as follows by Ferguson:

1. There is one secretarial position available.
2. The number n of applicants is known.
3. The applicants are interviewed sequentially in random order, each order being equally likely.
4. It is assumed that you can rank all the applicants from best to worst without ties. The decision to accept or reject an applicant must be based only on the relative ranks of those applicants interviewed so far.
5. An applicant once rejected cannot later be recalled.
6. The employer is very particular and will be satisfied with nothing but the very best. (That is, the payoff is 1 if the best of the n applicants is chosen and 0 otherwise.)

The goal is to maximize the probability of accepting nothing but the best secretary. The classical secretary problem has a remarkably simple solution:

Let the first n/e secretaries go by and then accept the first one that is interviewed and is better than all of those. The probability that this strategy selects the best secretary is approximately $1/e \approx 0.36788$. The mathematical evidence of this can be found in Ross [15] among a lot of others, and is discussed in chapter 2 of this thesis.
If these rules and this solution are used on the grab bag example, the probability that the highest valued cheque is accepted is $1/e$. The same theory can also be used to find the minimum value of a sequence.

In the last decades solutions are generated for plenty of variants on the secretary problem, such as a secretary problem where the rejection of a secretary may be delayed [14], or a problem where the payoff is not 0 or 1, but cardinal, meaning that the payoff is the exact value of the accepted secretary [4]. In some cases, of which one is given in section 1.1 by the problem at Statistics Netherlands, the mean or average of a sequence is important. In this paper, an approach is given to estimate the mean of a sequence as good as possible, without examining the full sequence. This approach is based on the mutual difference between successive relative means. Besides, the approach of the classical secretary problem is used in different ways to maximize the expected payoff of several of its variants. In several variants, the length of the frame, which is the observation phase before an applicant may be accepted, is optimized in relation to the expected payoff.
Two remarks on the variants of the secretary problem have to be pointed out. The first is that if the best applicant is observed in the frame, the last applicant of the sequence, and thus the last observed, must be accepted. The second remark is that only one applicant is accepted in all variants. Thus, if an applicant is accepted, all other applicants are rejected.

1.3 Research goals

The goal of the research described in this paper is to determine strategies that maximize the expected payoff. The expected payoff consists of profit and costs. Profits are payments based on the decision of the acceptance of an applicant or the estimation of the mean. The profit depends on the distance to the goal, which is the real maximum or mean of the sequence. Costs are fixed per observation necessary to make the decision. The main research goal is formulated as:

Maximize the expected payoff for stopping problems based on finding (one of) the highest values or approximating the mean of a sequence as good and as soon as possible.

Known strategies for variants on the classical secretary problem [7] are discussed in chapter 2. These form the basis for strategies described in this paper. The mean is estimated by comparing successive approximations of the mean, while numbers are observed one by one. To satisfy the goal, a simulation environment is created to test many variants on the secretary problem and the strategy for estimating the mean. Both procedures are based on information about observations, and a decision must be made only on this information. To validate the strategies it has to be known how good these strategies are. An optimal strategy is a strategy that has a maximum expected payoff at the moment that the observations are stopped, and - in case of the secretary problem - an applicant is accepted. An applicant is a person that wants the job as secretary, and in this paper it is a number in the sequence. This number denotes the rank or the specific value of the applicant. These type of problems is known as optimal stopping problems [8]. The strategies involved are called stopping rules.

1.4 Outline

In chapter 2, known solutions for secretary problems are described. Chapter 3 describes the model and the used approach with a simulation environment. Chapter 4 contains simulation descriptions with their results and discussion. Finally, general conclusions and an evaluation of the research goals are provided in chapter 5.

Chapter 2: **Known solutions**

In the introductory chapter several examples are given of secretary problems with a known solution. These solutions are discussed in short in this chapter. Firstly, the classical secretary problem, as stated by Ferguson [7], is discussed, which has a simple and remarkably accessible solution. In the subsequent section, the solution of a variant on the secretary problem with cardinal payoffs is described. In the third section, a variant where delay of accepting or rejecting an applicant is allowed, is discussed.

2.1 Classical secretary problem

In Seale and Rapoport [16] an approximation of the optimal solution of Gilbert and Mosteller [11] is given by examining what exactly happens while numbers are observed and decisions have to be made. The approach described in this section is strongly based on the description of the solution by Allen [2]. Consider a secretary problem, as discussed in chapter 1, with the six rules of Ferguson [7] and the objective to maximize the expected payoff by accepting the best secretary. The employer knows how many applicants may be observed, but does not know anything about their distribution, actual values, or relative ranks. The optimal strategy, consisting of observing a $1/e$ fraction of applicants and taking the first one who is better than each of those, is already given in chapter 1. Now it is derived analytically.
In a first attempt, it can be solved by observing short sequences. Suppose there are three applicants that want the secretarial position. One applicant has value 1, one applicant has value 2, and another applicant has value 3. The order in which they are observed by the employer is one of the following:

$$\{1, 2, 3\}, \{1, 3, 2\}, \{2, 1, 3\}, \{2, 3, 1\}, \{3, 1, 2\}, \{3, 2, 1\}$$

If without any strategy the first, second, or third applicant is accepted, the probability of success is 1/3. If the first applicant is observed but rejected, and the next one with a higher value is accepted, the second, third, and fourth permutation all lead to the acceptance of the best secretary, resulting in a probability of 1/2. The best applicant is accepted when observing and rejecting the first two applicants only in the first and third permutation, which again is probability 1/3. So the best strategy for three applicants is observing and rejecting the first and accepting one that is has a higher value than the first. The probability of getting the best one is 1/2.
Now suppose there are four applicants for the secretarial position. The possible permutations in which they are observed are:

$$\begin{array}{l}
\{1, 2, 3, 4\}, \{1, 2, 4, 3\}, \{1, 3, 2, 4\}, \{1, 3, 4, 2\}, \{1, 4, 2, 3\}, \{1, 4, 3, 2\},\\
\{2, 1, 3, 4\}, \{2, 1, 4, 3\}, \{2, 3, 1, 4\}, \{2, 3, 4, 1\}, \{2, 4, 1, 3\}, \{2, 4, 3, 1\},\\
\{3, 1, 2, 4\}, \{3, 1, 4, 2\}, \{3, 2, 1, 4\}, \{3, 2, 4, 1\}, \{3, 4, 1, 2\}, \{3, 4, 2, 1\},\\
\{4, 1, 2, 3\}, \{4, 1, 3, 2\}, \{4, 2, 1, 3\}, \{4, 2, 3, 1\}, \{4, 3, 1, 2\}, \{4, 3, 2, 1\}.
\end{array}$$

If the same strategy is used as in case of three applicants, rejecting the first applicant and accepting the next one with a higher value than the value observed at the first applicant, leads to success in permutations 5, 6, 8, 11, 12, 13, 14, 15, 16, 17, and 18. So, from the 24 permutations, 11 times the best secretary is accepted, which is a probability of $\frac{11}{24} = 0.45833$.

Rejection of the first two applicants and accepting the next best applicant in permutation 2, 3, 4, 8, 9, 10, 13, 14, 15, and 16 lead to the acceptance of the best secretary. Thus, in total this leads to 10 times the best secretary in 24 permutations, which is a probability of $\frac{10}{24} = 0.41667$. Rejection of the first three applicants is as good as the acceptance of the first applicant in the sequence, and the probability of accepting the best applicant equals $\frac{1}{4}$. If four applicants are available it is thus better to reject only the first one, because then the probability of getting the best is highest, namely 0.45833.
If this way of approaching is continued, the number of possible permutations will become very large if the length of the sequences increases. For that reason, an analytical method is needed.
Suppose n applicants are available and the employer wants to observe and reject m applicants before he accepts a next better applicant. The first m applicants are said to be in the *frame*. The frame is the observation phase before an applicant - whether candidate or not - may be accepted. A candidate is an applicant which contains the highest value of all observed applicants and thus may be accepted. There are two situations in which the interviewer can fail if he first observes and rejects all candidates in the frame: (1) the best secretary is observed and rejected in the frame, or (2) the best secretary is not in the frame, but preceded by an applicant which has a higher value than the applicants observed in the frame. Regarding the employers lack of information, the probability that any particular applicant out of n applicants has the highest value is $1/n$. If the first observed applicant after the frame, applicant $m+1$, has the highest value, she is accepted. So the probability that applicant $m+1$ is accepted and is the best, is $1/n$. The probability that applicant $m+2$ has the highest value is again $1/n$, but she is selected only if the applicant with the highest value in the first $m+1$ applicants is among the first m applicants (thus observed and rejected in the frame). The probability of this case is $\frac{m}{m+1}$, so the probability that applicant $m+2$ is the best and accepted is $\frac{m}{n(m+1)}$. The probability that the third applicant after the frame, applicant $m+3$, has the highest value is again $1/n$, but she will only be accepted if the applicant with the highest value in the first $m+2$ is among the first m. The probability is now $\frac{m}{(m+2)}$, so the probability that applicant $m+3$ is accepted and is the best is $\frac{m}{n(m+2)}$.
If this procedure is continued until the last applicant, n, is reached, the probability that applicant n has the highest value is $1/n$, but she is only accepted if the applicant with the highest value in the first $n-1$ is among the first m. The probability is $\frac{m}{n-1}$, so the probability that the last applicant is the best and is accepted is $\frac{m}{n(n-1)}$.
There is only one best secretary, so if the employer wants to accept the best secretary, he can do that with only one. The events that he selects one specific secretary are exclusive and their probabilities may be added [19]. The probability P of accepting the best secretary after observing the frame, is the following equation [2]:

$$P = \frac{m}{n} \times S, \qquad S = \left(\frac{1}{m} + \frac{1}{m+1} + \frac{1}{m+2} + \ldots + \frac{1}{n-1}\right) \tag{2.1}$$

The optimal value for m is the one which maximizes P, or in other words: the optimal frame length is one which maximizes the expected payoff. Allen [2] performed the experiments with the results given in table 2.1.

In the search for an optimal frame length - an optimal value of m - starting with $m = 1$, it is noticed that the probability of accepting the best secretary increases up to an optimal value. After the optimal value is reached, the probability decreases.
An analytical explanation is provided based on figure 2.1, where the graph of $\frac{1}{x}$ is shown and the area underneath this graph between m and n is linked with the amount $S = \sum_{i=m}^{n-1} \frac{1}{i}$ (see equation 2.1).

n = 3, m = 1, S = 1.500, p = 0.500 *
n = 3, m = 2, S = 0.500, p = 0.333

n = 4, m = 1, S = 1.833, p = 0.458*
n = 4, m = 2, S = 0.833, p = 0.416

n = 5, m = 1, S = 1.917, p = 0.416
n = 5, m = 2, S = 0.917, p = 0.433*
n = 5, m = 3, S = 0.583, p = 0.350

n = 6, m = 1, S = 2.283, p = 0.380
n = 6, m = 2, S = 1.283, p = 0.428*
n = 6, m = 3, S = 0.783, p = 0.392

Table 2.1: Experiments with equation 2.1

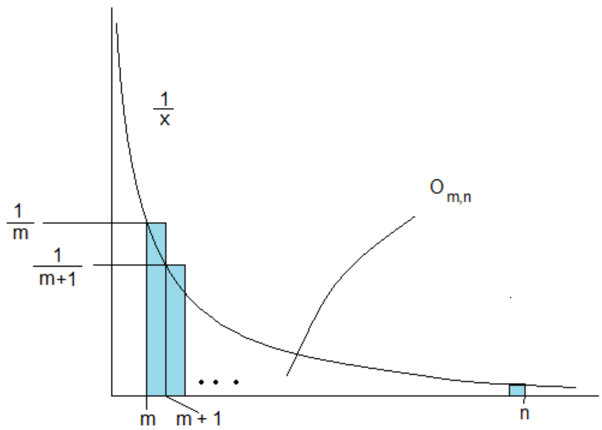

Figure 2.1: Integral for 1/x

$$O_{m,n} = \int_m^n \frac{1}{x} \mathrm{d}x = \ln n - \ln m = \ln \frac{n}{m}$$

Observe that

$$\sum_{i=m}^{n-1} \frac{1}{i+1} \leq O_{m,n} \leq \sum_{i=m}^{n-1} \frac{1}{i}$$

and from this it can be seen that

$$O_{m,n} \leq S \leq \sum_{i=m}^{n} \frac{1}{i} = \sum_{i=m-1}^{n-1} \frac{1}{i+1} \leq \frac{1}{m} + O_{m,n}$$

Hence, for large n and fixed fraction $\frac{m}{n}$, we have $S \approx \ln\frac{n}{m}$ and $P \approx \frac{m}{n} \times \ln\left(\frac{n}{m}\right)$. If $x = \frac{m}{n}$, then $P(x) = x\ln\frac{1}{x}$. Finding the optimal fraction corresponds to maximizing $P(x)$. This can be done by $P'(x) = 0$, where $P'(x) = \ln\frac{1}{x} - 1$, from which can be concluded that $x = \frac{1}{e}$. Thus for large n the optimal value of m is $\frac{n}{e}$.
For $n = 100$, the optimal frame length is then 37. For large n, using $m = \frac{n}{e}$, it is concluded that $P = 1/e$ which is 0.368, approximately. This proof can also be found in Ross [15].

2.2 Cardinal payoff

The objective of only being satisfied if the best applicant is accepted, may be too strict. If this strict demand is released, it can be seen that an employer would rather accept a higher valued applicant than a lower valued one, and not only be concerned with getting the best. Therefore in the cardinal payoff variant of the secretary problem, the payoff is the actual value of the accepted secretary, no matter if she is best or not. If the applicants of a sequence are drawn independently and identically from a uniform distribution, and the same strategy is used as for the secretary problem, the maximum expected payoff and the corresponding minimum number of applicants to reject are known from literature. As may be seen in the experiments, and what is known in literature, the optimal strategy is to reject the first $\sqrt{n} - 1$ applicants, and then select the first applicant which has a higher value than all observed applicants. Accepting this applicant maximizes the expected payoff of this variant. The proof is given in Bearden [4] and the experiments may be found in chapter 4.

2.3 Secretary problem with delay

Suppose the following situation. There are 100 applicants for a secretarial position. Every day exactly one applicant is interviewed. If an applicant is a candidate, she is not directly accepted when she leaves the interview room, but is told that she gets a response within 4 days. After observing the first applicant, this applicant is directly classified as candidate (no learning frame is applied here). Now the employer continues interviewing new applicants for 4 more days. If within these 4 days a higher valued applicant is interviewed, which is thus a candidate as well, then the first applicant is rejected, and the employer continues interviewing for at least another 4 days. Applicants with lower value than the value of the candidate are rejected. If a higher ranked applicant is interviewed in these 4 days, she is not accepted directly, but sent home with the same story that she gets a reaction within 4 days. The next three applicants are interviewed and if none is better, the one interviewed 4 days ago is accepted. With this strategy, a window of applicants with a candidate is created. The candidate is hired if no better applicant is found in 4 days. The window thus consists of the four applicants of the last four days. This is visually illustrated in figure 2.2. Suppose 4 is a candidate. The applicant with highest value in the window is only accepted if it is the highest value so far and if the reaction deadline of the employer is reached in the next step. If the window is slided three steps ahead (figure 2.2b), 4 is the last number of the window and will be rejected if the next applicant is observed. Because 4 is not the maximum of the window anymore (it is now 15), this one is rejected and the window moves forward. In figure 2.2c the applicant with value 15 is the next number to be rejected if it is not the maximum of the window. But because 15 is the maximum of the window, she is accepted as best secretary. What actually happens is that after an applicant is candidate, the next w applicants are observed. If no better applicant in the next w observations is a candidate, then the best applicant in the window is accepted. If the end of the sequence is reached, then the candidate that is in the window, is accepted.
Rocha [14] encountered an analytical solution for the problem described above. For uniformly

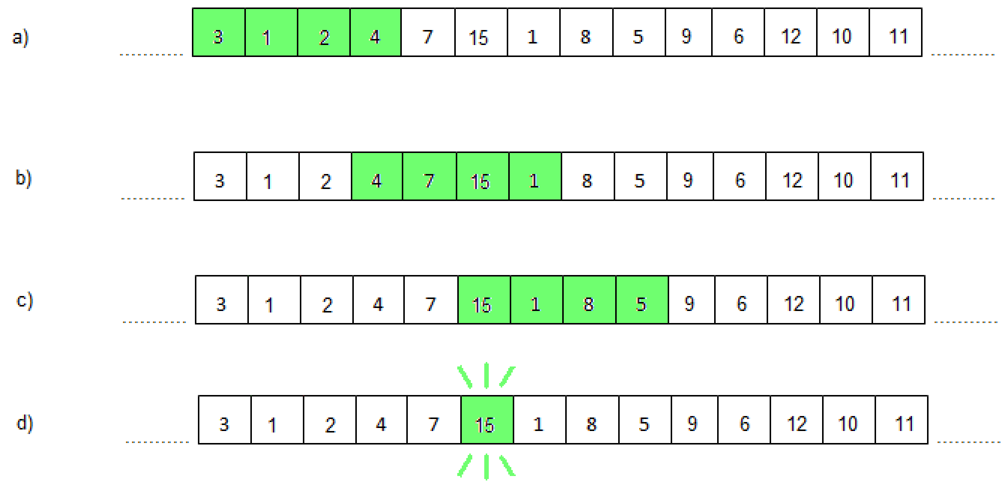

Figure 2.2: Example of rejection with delay

distributed sequences with increasing length n, Rocha determined an approximate solution for a delay window w, which is a fixed fraction α of the sequence length, i.e., $\alpha = \frac{w}{n}$. She obtained the maximum probability (p) of accepting the best secretary for $1/2 \leq \alpha \leq 1$:

$$p(\alpha) = 2 - \alpha + \ln \alpha \tag{2.2}$$

For $0 < \alpha < 1/2$, the maximum probability cannot be put in closed form. In chapter 4, simulations are described that define the maximum probability for cases where $\alpha < 1/2$ and cases where the distribution is normal instead of uniform.

Chapter 3: **Modeling**

A simulation environment is created to perform experiments with several parameter settings of the model. The simulation environment is built in Matlab. Matlab is chosen because it is a problem solving environment, designed for numerical computations. Sequences are analyzed in a straightforward, logical way and all variations described in the model can be implemented in a convenient way, partly with Matlab built-in functions.

3.1 Sequences

This thesis describes decisions that are made, based on the observation of numbers in sequences. Two types of distributions are used to generate the sequences: uniform and normal. The uniform distributions are generated by using the "randperm" function of Matlab, where the ranking of the applicants is considered, and the standard "rand" function, where only values are taken into account. Normal distributions are generated with the "randn" function. In order to compare the payoffs of uniformly distributed sequences and normally distributed sequences, both types need to contain the same maximum and minimum (especially for the cardinal payoff variant). In order to compare cardinal payoffs, all numbers are added with the minimum of the sequence and divided by the new maximum, so that the resulting minimum is 0, and the maximum is 1.

3.2 Payoff functions

In order to evaluate the strategies, payoff functions are used. The total payoff for accepting an applicant is the obtained profit for accepting that particular applicant, minus the costs for all observations needed for the acceptance of that applicant. The costs for every observation are identical and only based on the number of observations k. Thus total costs are $c \cdot \frac{k}{n}$, where $0 \leq c \leq 1$ and k is the number of observations k. In the next subsections, two payoff functions are described that are used in the model and in the experiments.

3.2.1 1/0 Profit

In the classical secretary problem "the employer is very particular and will be satisfied with nothing but the very best (that is, the payoff is 1 if the best of the n applicants is accepted and 0 otherwise.)" [7]. If a random applicant is accepted, the expected payoff - if $c = 0$ - is $1/n$. But if the strategy is performed which is described in section 2.1, the expected payoff of the secretary problem approximates $1/e$.
According to Ferguson's sixth rule, only the choice of the best secretary is worth 1 point. But imagine situations wherein the second best, third best or even one of the 10 best out of 10,000 is also a satisfactory result. For this reason the payoff function is extended with a profit for *one of the best* secretaries, which is also 1. Not only the secretary with highest value yields a profit of 1, but also a secretary that has at least threshold value

$$t(q) = -q \cdot (\max - \min) + \max, \quad 0 \leq q \leq 1. \tag{3.1}$$

The min and max in this expression are the maximum and minimum of the whole sequence. Thus, the employer is satisfied with an applicant that has at least value $t(q)$, and his profit is 1, if he selects one of those.
If for example the range of the sequence is [100, 200], and $t(q)$ is based on $q = 0.02$, $t(0.02) = -0.02(200 - 100) + 200 = 198$. The profit for an applicant that has at least value 198 is thus 1.
Another possibility is that the employer is satisfied with one of the best *ranked* secretaries. The threshold t in that situation depends on the rank of the applicants, instead of the actual value. Suppose that there are 10 applicants valued $\{0.5, 1, 1.5, 2, 2.5, 3, 3.5, 4, 9, 10\}$. If a secretary with rank t is satisfactory, for example $t = 3$, the applicant valued 4 is satisfactory, because she is ranked third out of 10.
In situations where the mean of the sequence has to be estimated, also the 1/0 payoff function is used. The estimated mean may have a relative error with respect to the actual mean of the sequence. If the estimated mean is within the thresholds set around the actual mean, the profit is 1, otherwise it is 0. A lower and upper threshold are determined from the relative error q of the estimated mean ($\overline{x}$) with respect to the real mean ($\overline{X}$). The relative error is:

$$\frac{|\overline{x} - \overline{X}|}{\overline{X}} \leq q \tag{3.2}$$

This results in an upper and lower threshold for the estimated mean. These thresholds are

$$(1 - q) \cdot \overline{X}$$

and

$$(1 + q) \cdot \overline{X}$$

respectively, where $q \geq 0$.

3.2.2 Cardinal profit

The second possibility in payoff functions is not to obtain profit 1 if the best or a sufficient secretary is assigned, but the profit is the actual value of the accepted applicant, independent of the fact whether she is the best or not. This is known as the cardinal profit. It is described in chapter 2 of this thesis, where also a strategy is described to maximize the total payoff if the distribution of the sequence is uniform, and $c = 0$.

3.3 Concepts

The strategies discussed in this paper may be seen as stopping procedures. A moment is determined to stop observing applicants and accept one or to estimate the mean of the sequence. The payoff at that moment is expected to be maximal. All strategies described here about maximizing the expected payoff are based on first observing a frame and then making a decision in every next observation. In estimating the mean of a sequence, the strategy is based on the observation of the fluctuations of consecutive means for some number of times. In this section, the important properties and variables of the strategies are discussed. First two concepts are described which are used in the approaches for accepting the best applicant or estimating the mean. After that, short descriptions are given for the possible strategies and variations in the experimental setup.

3.3.1 Frame

In the strategies used in this thesis, first a frame is observed, and then for every next observed applicant a decision is made. The length of the frame is the minimal number of applicants to observe, before a decision may be made. The frame is the learning phase, where applicants are observed and the highest value, lowest value, and mean are remembered. If for example an applicant that is observed after the frame has a higher value than all applicants observed so far, she is accepted. Otherwise she is rejected. The length of the frame, m, should be chosen in an optimal way.
According to the second rule of the classical secretary problem the number of applicants n is known. Frame length m is based on the number of applicants in a sequence, and is the main parameter to vary in strategies in this research. As an example of frame length, regard the classical secretary problem. The optimal length of the frame is n/e. If costs are increased in the payoff function, for example if c increases from 0 to 1 in the payoff function, m is expected to decrease.
Because n is known in all situations described in this paper, the length of the frame can be expressed in relation to n.
Although there are many papers that deal with an unknown number of candidates [1, 5], the rule is not varied in this research.

3.3.2 Window

As is explained in chapter 2, variants on the secretary problem are known where an observed applicant is not accepted directly, but this decision is made after another w applicants are observed. In the situation of the paper of Rocha [14], there is no frame to be observed first. The first applicant of the sequence is observed, and if no better is observed in the next w, which is referred to as the window, she is accepted. It may thus be said, that the window is a movable frame, and is the number of stages that the decision maker can wait after seeing an applicant to accept or reject her. The window is also used in estimating the mean of a sequence. If consecutive means are observed and they are in some sense "stable" (see section 3.4.2), a decision may be made to stop. But if the mean is not yet stable, more numbers are observed. The window is thus a decision phase; a number of applicants that must be observed after a candidate is observed or a mean is probably estimated.

3.4 Experimental setup

The concepts and payoff functions described in this chapter result in a lot of possibilities for variants on the secretary problem and estimation of the mean. Some of these variants, which are of practical and scientific interest, are simulated and described in chapter 4. In this section, the experimental setup is described, and strategies are discussed. Generic information that applies to all tests is outlined.
Experiments are performed in order to evaluate strategies. The evaluation is done by observ-

ing the influence of variations on the strategies on the payoffs. Optimal frame lengths m and optimal window lengths w are determined, for which the expected payoff of the strategy is maximal.
All experiments contain a number of tests. These tests are different in their frame length or window length. The tests are performed with a certain number of runs. The number of runs is for all tests 10,000. Every run is the execution of the strategy for one sequence, and all sequences are of length $n = 10,000$, unless otherwise stated. The time consumption is linearly dependent on the number of applicants in a sequence and the number of runs. The observation of 1000 sequences with 10,000 applicants takes 260 seconds (on a dual core PC with 2 Ghz internal memory), which means that 10,000 runs on sequences with 10,000 applicants in a test are expected to last approximately 2600 seconds. Most experiments contain tests with 20 different frame lengths or window lengths. The tested window lengths are determined after a first observation with fewer runs, to observe where interesting differences in results may happen. Because many of the frame and window lengths in the actual test are smaller than 50% of the sequence, the execution time of most runs is shorter. The observation of 1000 sequences with $n = 5000$ for example, takes 61 seconds, which results in an expectation of 610 seconds for 10,000 sequences. The overall time for an experiment is then expected to be not more than 1,5 hour in total. This time consumption is satisfactory in a way that experiments can be performed in reasonable time.
Results differ if new random sequences are chosen for every test. The variance and fluctuation in payoffs in different frame lengths are high. If the same random sequences are chosen for every separate test, more satisfying results are obtained. The difference can be seen in figure 3.1, where expected payoffs are plotted for different frame lengths in case of the classical secretary problem. The plot which shows the expected payoffs for different frame lengths for random sequences that are new for every test, is shown in figure 3.1a. The expected payoffs for random sequences which are the same for every frame length test are shown in figure 3.1b. To obtain the same pseudo-random sequences for each test, a seed value in Matlab is used. This means that for each test the pseudo-random generator is set to a fixed value.

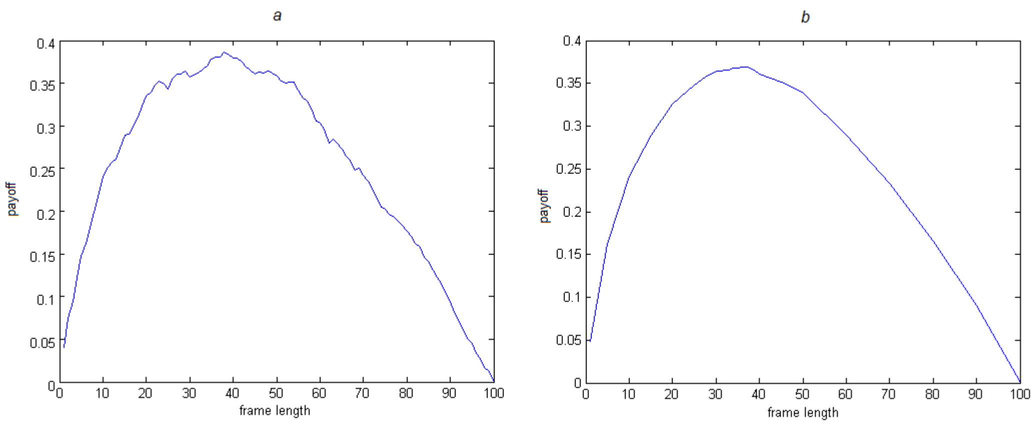

Figure 3.1: Classical secretary problem (a) without seeded random values and (b) with seeded random values

The profits in all experiments are based on the actual value of the accepted applicant. The best applicant is thus the applicant with the highest value. If a deviation is allowed in the maximum of the sequence, $t(q)$ is based on the value, not on the rank. Since tests with sequence maximum deviation $t(q)$ are performed with uniformly distributed sequences with values between 1 and 10,000 the actual value of an applicant is directly related to her rank and there is no difference in relating the payoff to the actual value or the rank of the applicant.

3.4.1 Secretary problems

The strategy used for secretary problems is described here in short, because it may be known from other sections already. In all variants of the secretary problem, described in this thesis, first a frame is observed. After that, a decision has to be made about every next applicant. If no window is allowed, the observed applicant is rejected or accepted directly after observation. If a window is allowed, the observed applicant is accepted or rejected after observing the next w applicants.

3.4.2 Estimating mean

In this research the mean of a sequence is to be estimated in as few observations as possible. A maximization of the probability for a close estimation of the mean of a sequence, while just a fraction of the sequence is analyzed, depends on the difference in consecutive means and the window length w. Optimal values are determined with simulation runs.
The estimation of the mean is performed by determining the mean of observed numbers. This mean value is determined for every new observed applicant. The mean of the first k observed numbers is determined, followed by the mean of $k+1$, $k+2$, etcetera. The mean value becomes more stable if k grows larger. Stable in this case means that the mean of k is close to the mean of $k-1$, which is close to the mean of $k-2$, etcetera. When k grows larger, the mean is becoming more stable. This can be seen in equation 3.3.

$$\overline{x}_{k+1} = \frac{k \cdot \overline{x}_k + \lambda_{k+1}}{k+1} \tag{3.3}$$

In this equation $\overline{x}$ is the mean, and λ is the value of the current observation. The exact mean of a whole sequence is found if $k = n$, thus when all numbers are observed. But analyzing all applicants costs time and money, so in several cases it may be important to estimate the mean value of a sequence with observing as less applicants as possible.
Figure 3.2 shows an example for larger sequences, where the mean varies, but is getting more stable if k increases.

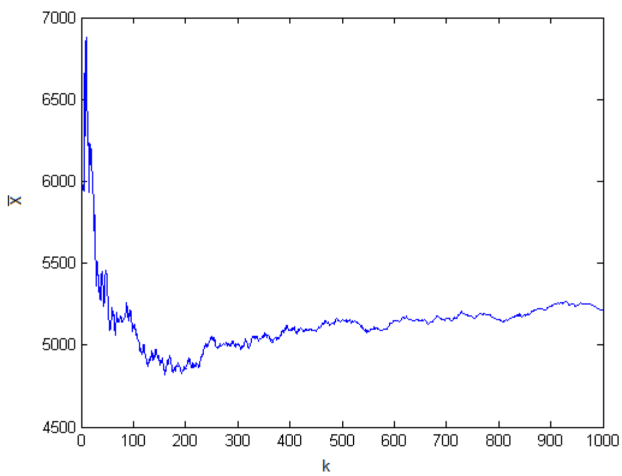

Figure 3.2: Stability of the mean

Chapter 4: **The experiments**

In the previous chapters the possibilities of the variants on the secretary problem used in this research are described. Also the variants in determining the mean of a sequence are described. Not all variants are of practical and scientific interest, and some interesting simulations and results will be discussed in detail in this chapter. Simulations are performed for situations with known results, as in the variants of chapter 2, to validate and verify the approach used in this thesis. Next to that, strategies are tested whose results cannot be analyzed in a simple mathematical way. In the first part of this chapter, several variants on the secretary problem are described and results are shown and discussed. Thereafter the strategy for estimating the mean of a sequence is simulated and compared with known statistical theory.

4.1 Secretary problems

In this section five experiments are outlined in which the payoff or the type of strategy is different from the classical secretary problem. The simulation tool generates results for the experiments of cases that cannot easily be solved by mathematical analysis. In the first experiment, the secretary problem with delay is simulated. The results of Rocha [14] are validated and extended for $\alpha < 1/2$. Besides, a comparison is given for normally and uniformly distributed sequences. In the second experiment, the cardinal payoff function is tested. Known results of the paper of Bearden [4] are simulated to validate the results of the simulation, and a comparison is given for uniformly and normally distributed sequences. In the succeeding two experiments, the exact rules of the classical secretary problem are obeyed, but the payoff functions are different. In the last experiment of the secretary problems, the influence of a different strategy is observed.

4.1.1 Acceptance with delay

The situation where delay is allowed in the rejection or acceptance of an applicant is outlined in section 2.3. The paper of Rocha [14] encounters an analytical solution to this problem for $1/2 \leq \alpha \leq 1$, and uniform distribution. The frame length in this case is 0. In this experiment the problem is simulated for situations where the sequences are uniformly distributed and $0 < \alpha < 1/2$, and for normally distributed sequences where $0 < \alpha \leq 1$. Another experiment is set up where the profit is 1 if the accepted applicant has at least threshold value $t(q)$, in order to observe the difference in uniformly and normally distributed sequences. The costs in both experiments are 0.

Experiment 1

This experiment is set up to observe the payoffs of uniformly distributed sequences and normally distributed sequences for the "acceptance with delay" variant. The actual payoffs, observed while performing the experiment are compared to the theoretic value of Rocha.
The value of α varies in every test: $0 \leq \alpha \leq 1$. The payoff function is 1 for accepting the best secretary, 0 otherwise. Frame length is 0, and $c = 0$. The maximum probability of accepting the best secretary for different values of α is plotted.

Results and discussion

The expected probability of finding the best secretary for different values of α is shown in the plot of figure 4.1. The red line in figure 4.1 is the function $f(\alpha) = 2 - \alpha + \ln\alpha$ adressed

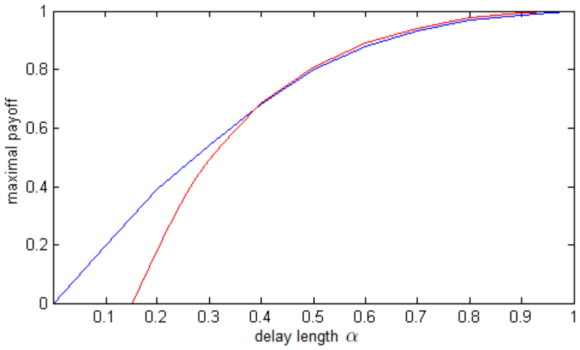

Figure 4.1: Results of accepting an applicant with delay

by Rocha [14]. The blue line is the result of the simulation for values of α. Both uniformly and normally distributed sequences obtain similar results for the maximum probability of accepting the best secretary, and are united in the blue line. Regard that the expected payoff for $\alpha = 0$ is not 0, but $1/n$, because the first observed applicant must be accepted and her probability of being the best is $1/n$.
It may be observed that the theoretic approach of Rocha [14] produces negative probabilities if $\alpha < 0.16$. If $\alpha \geq 1/2$ the results of the experiment are similar to the theoretic value. This confirms that the simulation environment generates feasible results for $0 < \alpha < 1/2$. These results have not yet been analyzed mathematically, but this experiment provides outcomes for the cases where $\alpha < 1/2$.

Experiment 2

In experiment 1 no difference is observed in payoff between uniformly and normally distributed sequences. If the payoff function however is modified in a way that an accepted applicant valued at least $t(q)$ has also profit 1, it is expected that there is a difference in payoffs between uniformly and normally distributed sequences, when a delay of α is allowed. Because values of applicants in normally distributed sequences are expected to be concentrated around the mean of the distribution, less applicants obtain profit 1 than in a uniformly distributed sequence, where all values are equally likely to be in the distribution.
Tests are performed where the threshold $t(q)$ for obtaining profit 1 is based on $q = 0.0005$ and $q = 0.001$.

Results and discussion

For situations where a delay α is allowed and the profit is 1 for accepted applicants with at least value $t(q)$, plots are shown for $q = 0.0005$ and 0.001 in figure 4.2. As is expected, there is a difference between uniformly distributed sequences (blue lines) and normally distributed sequences (red lines).

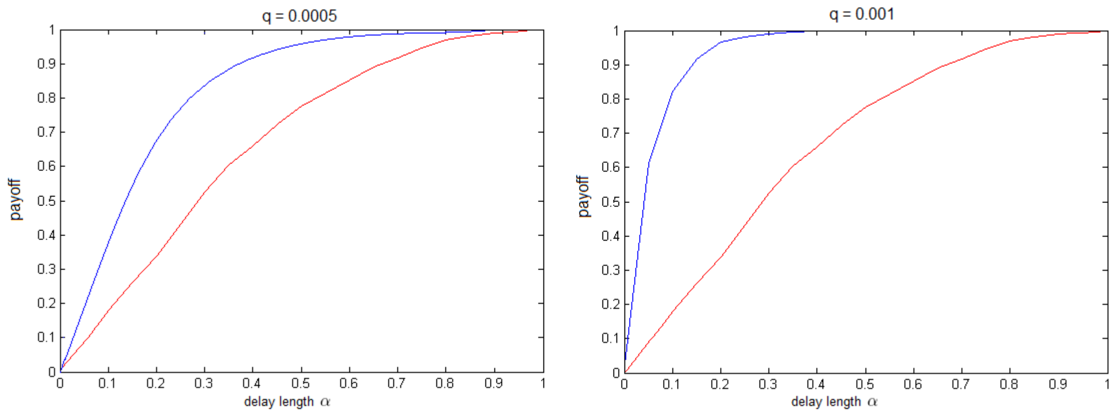

Figure 4.2: Results of accepting an applicant with delay, deviation in payoff

4.1.2 Cardinal payoff

The paper of Bearden [4] shows an optimal frame length for the probability of maximizing the cardinal payoff of an accepted applicant. The type of strategy for accepting an applicant is the same as for the classical secretary problem. To observe the difference in uniformly and normally distributed sequences when the payoff is cardinal, an experiment is set up that first validates the simulation environment to the theoretic value of Bearden. Uniformly distributed sequences are the input for this validation, and different frame lengths are used for each test. The frame length with the highest payoff is observed and is expected to be $\sqrt{n}-1$. After that, it is observed whether there is a difference in optimal frame length if normally distributed sequences are the input for this strategy and this payoff function.

Experiments

Two experiments are performed here. The first experiment is the validation of the simulation environment with respect to the theory of Bearden [4]. In the experiment, three tests are performed. The tests are different in their sequence length n. The tests are performed for $n = 1,000$, $n = 5,000$, and $n = 10,000$. The respective optimal frames for uniformly distributed sequences according to Bearden are 30.6, 69.7, and 99, which is 3.1%, 1.4%, and 1% of the sequence length respectively. For each test, the payoff is the actual value of the accepted applicant. The frame length is varied for each test, to observe what frame length gives the highest payoff.
The second experiment is the input of normally distributed sequences, with the same lengths for n as in the first experiment, and the same payoff function. Just as in the first experiment, the frame length is varied. There is no theoretic result for this case.

Results and discussion

If the experiment is performed for sequences where $n = 1,000$, the theoretic optimal frame length by Bearden is $\sqrt{1,000} - 1$, which is 30.6. The payoffs for all frame lengths from 1 to 10,000 are given in figure 4.3. The blue line represents uniformly distributed sequences, the red line normally distributed sequences. It may be observed that the frame length of the maximal payoff is indeed 3.1% of n, in case the distribution of the sequence is uniform. For larger n, the optimal frame length is smaller in relation to n. For this reason, and because it is known by observation that the payoff decreases shortly after the optimal frame length

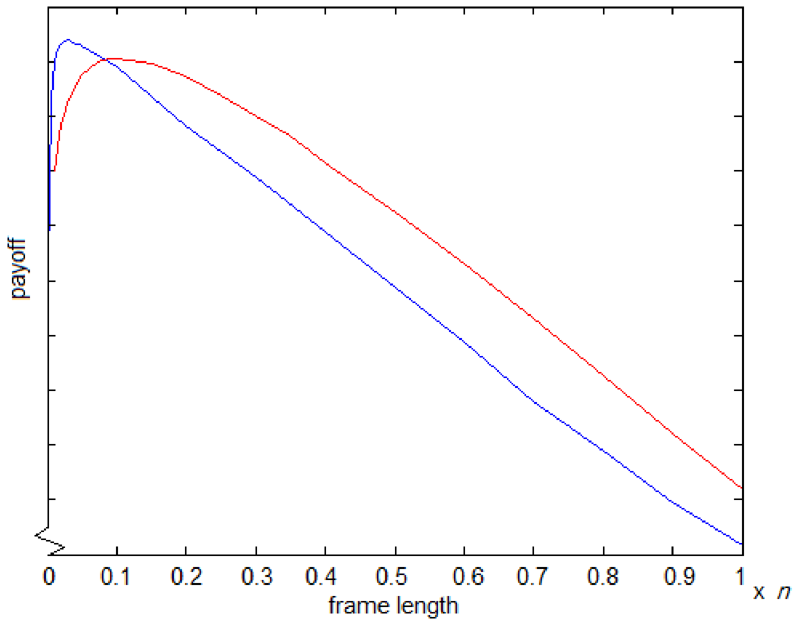

Figure 4.3: Results for cardinal payoff variant, n = 1,000

is reached, only the region around the maximal payoff and optimal frame length is shown in the next tested situations. The same applies for the normally distributed sequences, where the maximum payoff is observed to be around 9% of n for $n = 1,000$. Observe that in the figures of this subsection the plots have a range not starting at zero; this is indicated by the broken y-axis.
The results are shown in figure 4.4.

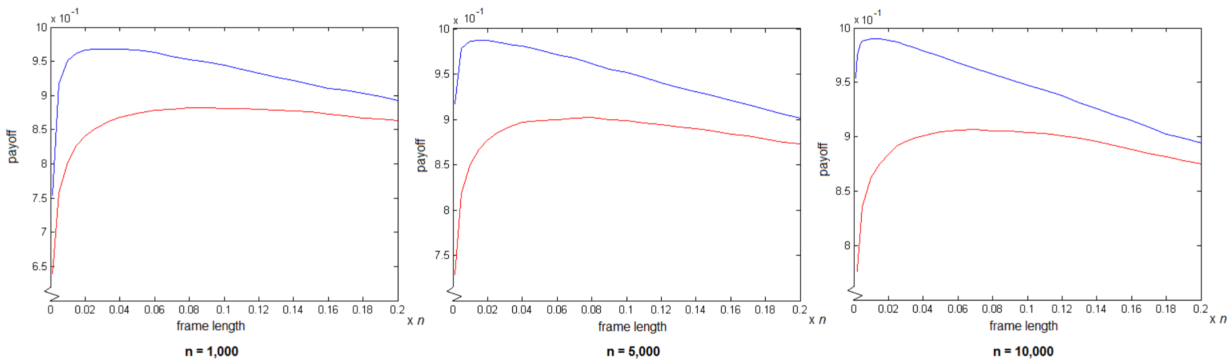

Figure 4.4: Results for cardinal payoff variant

In the tests with uniformly distributed sequences, the theoretic results of Bearden that are given in the previous subsection are obtained. The plots are smoothly approaching the optimal frame length. After passing the optimal frame length, the plots indicate lower payoffs. In the first figure, where $n = 1,000$ the decrease in payoff seems to be insignificant, but this depends on the used scale. As is shown in figure 4.3 the payoff gradually decreases at the right side of the optimal frame length.
If normally distributed sequences are tested, as is performed in the second experiment, it

is observed that the size of n does not influence the optimal frame length as much as in uniformly distributed sequences. The frame length that belongs to the maximal payoff for normally distributed sequences decreases from 9% of n for $n = 1,000$ to 7% for $n = 10,000$. Larger values of n would probably show that the optimal frame length decreases slowly to shorter frames with respect to n, but can not be tested because of the long execution time within Matlab. It is interesting for future research to see if the optimal frame lengths are similar for uniformly and normally distributed sequences for very large n.
The actual payoffs for uniformly distributed sequences are higher than for normally distributed sequences. The reason is explained with the help of figure 4.5. The probability

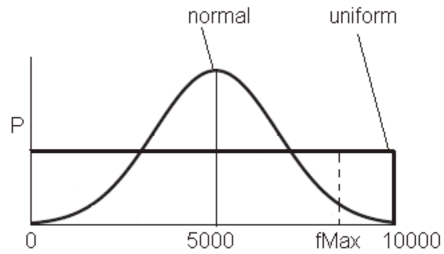

Figure 4.5: Normal and uniform distribution

that in the frame of a uniformly distributed sequence a high value is the observed maximum (fMax), is higher than in case of a normally distributed sequence. Since after the observation of the frame, a higher valued applicant must be accepted, the probability of a high valued applicant in a uniformly distributed sequence is higher than in a normally distributed sequence. Because the payoff in this variant only depends on the actual value of the accepted applicant, the uniform distributed sequences obtain a higher payoff. A similar explanation applies for the reason that the payoffs are higher if n increases.

4.1.3 1/0 profit with costs

The optimal strategy of the classical secretary problem aims for accepting the best secretary, but does not care about how many interviews are necessary to get the best secretary. In real life situations there are often costs involved in the observation (interview) of each applicant. The costs for observing an applicant are subtracted from the profit of accepting the best secretary. This profit is similar to the classical secretary problem: 1 if the best secretary is accepted, 0 otherwise. In this experiment different values of frame lengths are tested to observe for which frame length the expected payoff is maximal if c increases. An example of the payoff function is outlined here in order to make clear how it exactly works. Suppose there are 1,000 applicants, and the costs for observing an applicant are $c = 0.001$. If the 400^{th} applicant is accepted and is indeed the best applicant, the payoff is $p = 1-(400\times0.001) = 0.6$. If the 400^{th} applicant is accepted and is not the best, the payoff is $p = 0-(400\times0.001) = -0.4$. If the last applicant in the sequence is accepted and she is not the best, then the payoff is $0 - (1000 \times 0.001) = -1$.

Experiment

In this experiment five different values for costs c are tested and observed is whether there is a link between the frame length, maximal payoff, and costs. The tests (every different value for c is a separate test) are different in frame length: from 1 to n. The sequences are uniformly distributed, in order to compare this experiment with the classical secretary problem. The five tests are only different in cost value c, which are 0, 0.1, 0.25, 0.5, and 0.75 respectively. The test with cost value 0 is performed for validation, because costs 0 should give the same results as the classical secretary problem.

Results and discussion

The results for the five different cost values are shown in figure 4.6. In the first situation,

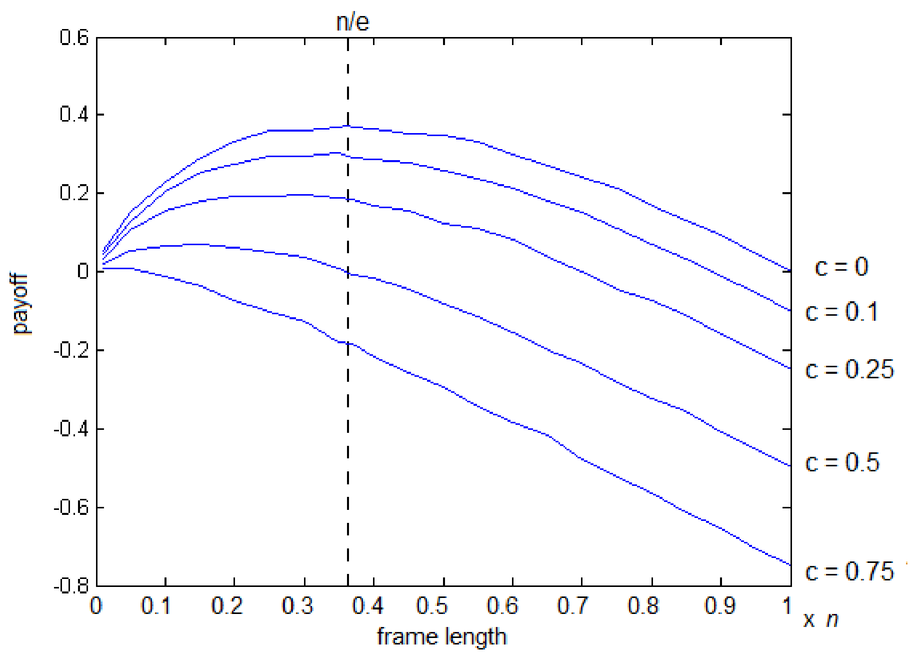

Figure 4.6: Results for 1/0 with costs payoff function

where $c = 0$, the optimal frame length is approximately n/e. The corresponding maximal payoff is $1/e$ (≈ 0.37). These are the optimal frame length and maximum expected payoff of the classical secretary problem. In the next plots, where costs are increasing, the optimal frame length moves to the left, thus to a smaller frame. The payoffs are lower. If $c = 0.75$, the frame has to be short. This is because the costs are relatively high compared to the payoff. If no applicant is accepted, the payoff is -0.75 which has significant influence on the total payoff.

4.1.4 Sequence maximum deviation

In the next experiment again the same rules and the same strategy are used as for the classical secretary problem. The payoff function however is different. Not only the acceptance of the best applicant obtains a profit of 1, also the acceptance of an applicant with at least threshold $t(q)$ obtains a payoff of 1. The acceptance of all other applicants result in payoff 0. There are two interesting procedures to approach this problem. In the first approach, the frame length is the same as in the experiment of the classical secretary problem (namely n/e), and q is increased. In the second approach it is tested what size the frame needs to be in order to maximize the payoff for this strategy and this payoff function if q increases.

Experiment 1

In this section, an experiment is performed to observe what the effect on the payoff is, when the frame length is n/e and q increases from 0 to 1 (which means that threshold $t(q)$ decreases). The distribution of the sequences in this experiment is uniform, in order to compare the results with the classical secretary problem.

Results

The results are shown in figure 4.7 and 4.8. Figure 4.8 is an enlargement of the left side of figure 4.7. It may be observed that the increase of q affects the expected payoff significantly in the beginning (where $0 < q \leq 0.005$). Right from $q \approx 0.005$ the expected payoff increases slower with the allowed sequence maximum deviation. The explanation of the strong influence of a little deviation is discussed here. In the classical secretary problem, situations may happen that for a sequence with numbers $\{1, 2, 3, ..., 98, 99, 100\}$ the maximum of the frame is 97. If the strategy is performed, there may be a chance that the applicant with value 98 or 99 is assigned. In the classical secretary problem, the payoff would be 0, because the employer is only satisfied with nothing but the very best. In the situation of allowed sequence maximum deviation, for example $q = 0.05$ and thus $t(q) = 95$, accepted applicants with value 98, 99, and 100 give a profit of 1. The influence of higher values for q are not that extreme. Thus the linear part of the plot ($q > 0.005$)) may be declared as follows: if the percentage of allowed deviation from the sequence maximum increases, also the number of applicants that give profit 1 increases. This increase in number of applicants results in the observed linear increase of the expected payoff, and ascends slowly to an expected payoff of 1.

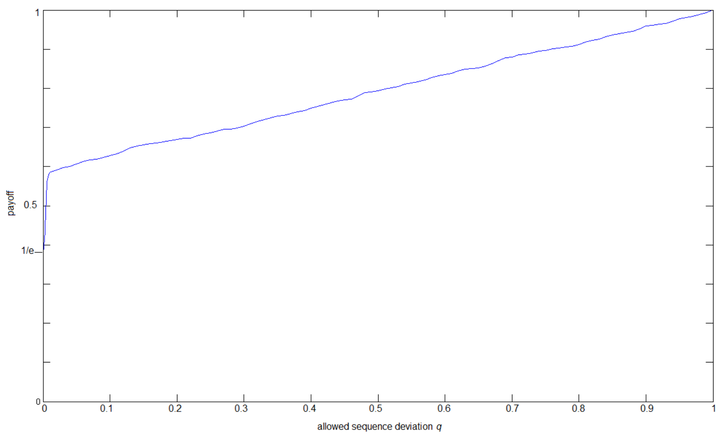

Figure 4.7: Results for increasing q, frame length $= n/e$

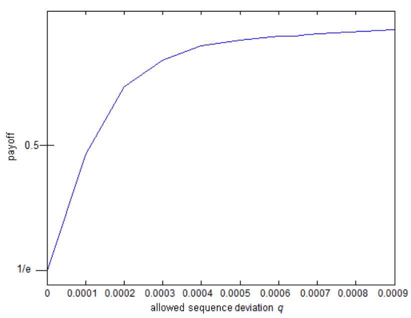

Figure 4.8: Results of increasing q for $0 \leq q \leq 0.0005$

Experiment 2

In the second experiment, for every value of q between 0 and 1, the frame length with the highest expected payoff is the optimal frame length. Tests are performed for $0 \leq q \leq 1$ and frame lengths between 1 and n. Thus, more clarified, for every value of q, the optimal frame length is plotted.

Results and discussion

The outcome of this experiment is the optimal frame length, belonging to a specific value of q. If q increases, the length of the optimal frame decreases, and the expected payoff increases. It may be observed that if $q = 1$, all applicants are satisfactory and the payoff is 1 for every applicant, independent of the frame length.
If the optimal frame length is plotted against q, the result is the graph in figure 4.9. The explanation of the decreasing frame lengths, is that by increasing q, the number of applicants that obtain profit 1 also increases. A shorter frame suffices in these situations, because the frame maximum may be lower if an applicant must be accepted which has a higher value than this low valued applicant. It may be observed that the optimal length of the frame

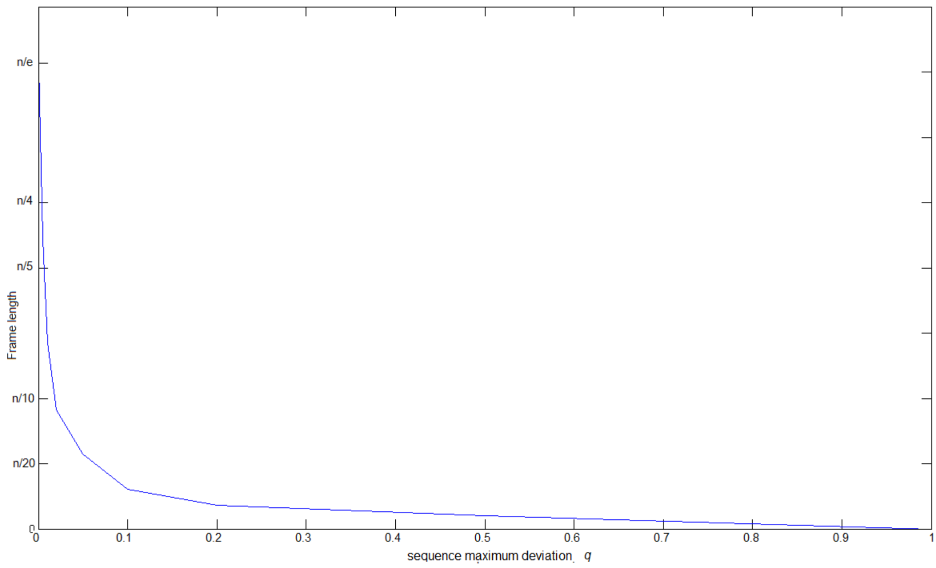

Figure 4.9: Optimal frame length plotted against allowed sequence maximum deviation q

is decreasing rapidly if the allowed sequence variation grows, and then slowly approaches a frame of length 0, if q is 1. The explanation is given in the previous experiment.

4.1.5 Frame maximum deviation

In the next experiment, the rules of the classical secretary problem are obeyed, and the payoff function is similar to the payoff function of the previous section. The strategy however is different. If in the classical secretary problem the maximum of the sequence is already observed in the frame, all applicants are observed and the last applicant is accepted. This may bring a lot of costs and does not guarantee a high value of the accepted secretary. In order to 'guarantee' the acceptance of a high ranked secretary, another strategy is simulated. After observing the frame, not an applicant with a higher value than the frame maximum is accepted, but one that has at least threshold value

$$s(r) = -r(\text{fMax} - \text{fMin}) + \text{fMax}, \quad 0 \leq r \leq 1,$$

where fMax is the maximum observed in the frame and fMin is the minimum observed in the frame. With this more loose restriction the probability that an applicant is accepted before the end of the sequence is reached, is larger. Thus, if the frame is observed, the next applicant that has at least value $s(r)$ is accepted.
It may be obvious that in this case most of the times not the best applicant is accepted. The variation of r with respect to the maximum of the frame must thus be combined with an allowed variation of q related to the maximum of the sequence. Summarized, an applicant is only accepted if she has a value that is at least $s(r)$, and has profit 1 if she has at least value $t(q)$. Otherwise the profit is 0. Thus, suppose the maximum of a sequence is 100 and the minimum is 0, the maximum of the frame is 98, $s(r)$ is 94, and $t(q)$ is 95. If after the frame an applicant with value 97 is observed, she will be accepted, because $97 > 94$. The payoff for assigning this applicant is 1, because $97 > 95$.

Experiments

In order to test this strategy, two values are set: $s(r)$ and $t(q)$. The total payoff of each test depends on $s(r)$, $t(q)$, and the frame length. Similar to former simulations, the payoffs are maximized as function of the frame length. The tests are executed for frame variation values $r = 0$, $r = 0.005$, and $r = 0.01$, and for maximum variation values $q = 0$, $q = 0.005$, and $q = 0.01$. Costs $c = 0$ in this experiment.

Results and discussion

In figure 4.10 the results for the simulations of the cases wherein the accepted applicant must have at least a value $s(r)$ with respect to the maximum of the frame, are shown. The plots are shown in a table format; the plots in the left column are based on accepted applicants which have at least value $s(r)$, where $r = 0$, the center plots $r = 0.005$, and the plots on the right $r = 0.01$. The division in rows is for payoffs of 1 if the applied secretary has at least value $t(q)$, where $q = 0$, $q = 0.005$, and $q = 0.01$ respectively for the top, center, and bottom row. In the top left plot there is thus no frame maximum deviation and no sequence maximum deviation allowed, which means that this case is similar to the case where no variation is allowed: the classical secretary problem. The plots in the left column are similar to the situations in the previous section; the strategy of the classical secretary problem is performed, but the payoff is 1 not only for the best secretary, but also for one of the best. The optimal frame length decreases if the allowed deviation with respect to the sequence maximum increases. The total payoff increases as well. If r increases in the top row, the optimal frame length is around $n/2$, but the payoff is much lower than in the classical secretary problem. This is because in the top row only the best secretary has payoff one, although the probability that this one is chosen, is much smaller here.
In three of the remaining four plots, the maximal payoff is close to 1 and the plots are in a bow shape, which indicates that a variation in the frame maximum and the sequence maximum leads to similar payoffs. The reason for this is that the probability of the acceptation of a satisfactory applicant is high because there are more applicants that lead to payoff one, and the chance that they are accepted is also high, because of the difference in the strategy. The reason that the plot with $r = 0.01$ and $q = 0.005$ shows a lower payoff, is more or less the same as for the top row plots; applicants may be accepted that have lower threshold value $s(r)$, but the threshold with respect to the sequence maximum is higher. It may thus be observed that there is a strong relation in $t(q)$ and $s(r)$.

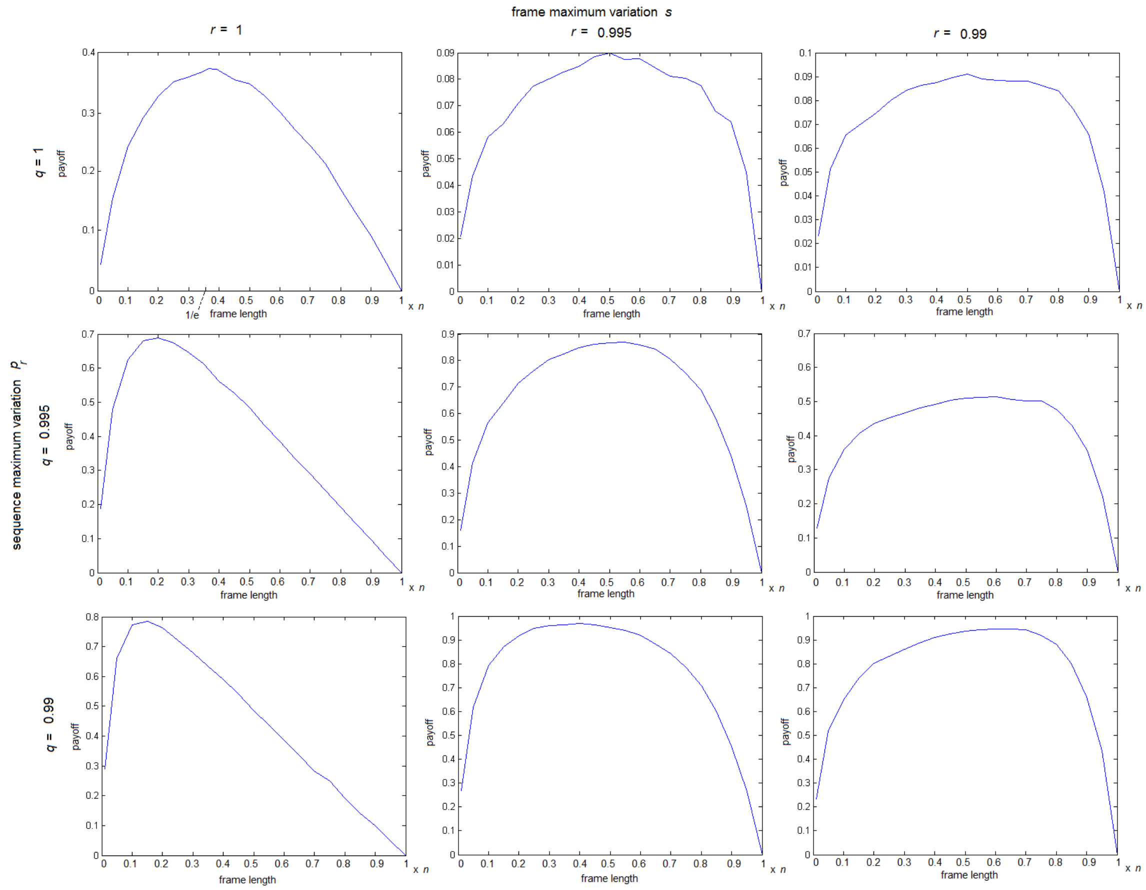

Figure 4.10: Results for allowed frame maximum variation

4.2 Mean

In the preceding chapters it became clear that optimal stopping may also lead to a decision of estimating the mean value of a sequence as good and as soon as possible, while the applicants in that sequence are observed one by one. For every kth applicant that is observed, a new mean is calculated and it is compared to the mean of the applicants $k-1, k-2, ..., k-w$, where w generates a window of mean values to compare with and to determine whether the mean is stable. To determine whether a mean is stable, the term stable needs to be measurable. Therefore two parameters are necessary, that determine the term stable: *maxmean variation* d and *window length* w. The first term, maxmean variation, is the maximum relative difference of two mutual means to each other, wherefore it can be concluded that the means have a similar value. As explained before, the larger k becomes, the more the mean tends to go to a fixed value. After it is determined that the mean value after observation k is not varying too much by being within the thresholds $(1-d) \cdot \overline{x}_k \leq \overline{x}_{k-1} \leq (1+d) \cdot \overline{x}_k$ for two consecutive means, it may not be concluded that the mean is stable and thus approaches the real mean. For that reason window length w is important. It determines the number of successive times that the mean value is within the thresholds. If the mean value of two consecutive observations is within the thresholds, this has to be so for the next $w-1$ observations.

The payoff function is based on the 1/0 profit function. The combination of the two parameters and the parameter q of the payoff function is tested. A comparison is made between the number of observations needed according to existing statistical literature and the number of observations needed for the strategy performed in this thesis.

Two payoff functions are experimented: one without costs and one with costs. In the no costs case, tests are performed with normally distributed sequences and with uniformly distributed sequences, to see if there is a difference in how many observations are necessary to estimate the mean.
In the experiment with costs, the payoff is 1 for estimating a mean that is within the thresholds, but costs are subtracted for every observed applicant. In the experiment without costs, w and d are varied. Costs $c = 0$ in this experiment.

Experiment 1

In this experiment parameters d and w are varied, in order to guarantee the estimation of a mean with maximal error with respect to the real mean. In order to get a good estimation of the mean $\overline{X}$ of a sequence, the relative error allowed with respect to the real mean is 5%. For this strategy this means that a profit of 1 is obtained, if

$$0.95 \cdot \overline{X} \leq \overline{x} \leq 1.05 \cdot \overline{X}, \tag{4.1}$$

It is interesting to know how many observations have to be done, to get an expected payoff of 0.95. In this experiment, three values for d are tested: 0.001, 0.003, and 0.005. The window length is varied for each test.

Results and discussion

The payoff increases if the length of the window increases, as can be observed in figure 4.11. But if the window increases, more observations are necessary to get an expected payoff of 0.95. Suppose for example $w = 10$ and the mean of observed applicant k is within the thresholds of the preceding mean $k - 1$ for eight successive times. The ninth time the mean is not within the thresholds, and again at least 10 applicants must be observed. In that case the number of observations increases rapidly.
In figure 4.11 a difference in normally (red) and uniformly (blue) distributed sequences can be observed. The reason of the higher payoff for normally distributed sequences is that the value of applicants in the normally distributed sequences have more probability to be close to the mean value, as can be seen in figure 4.5.

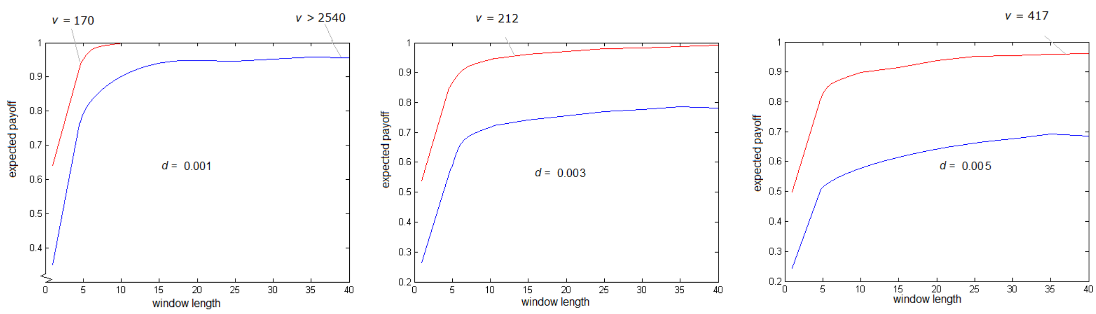

Figure 4.11: Prediction of mean

Uniformly distributed sequences have more fluctuations in their consecutive means and need more observations to be stable and obtain an expected payoff of 0.95. As can be seen in figure 4.12 the moment of stopping is not the moment that the mean is within the 5% error threshold (grey area in the figure), and thus more applicants must be observed. The window in this situation is $w = 10$. If w is increased, many more observations are needed. It may be

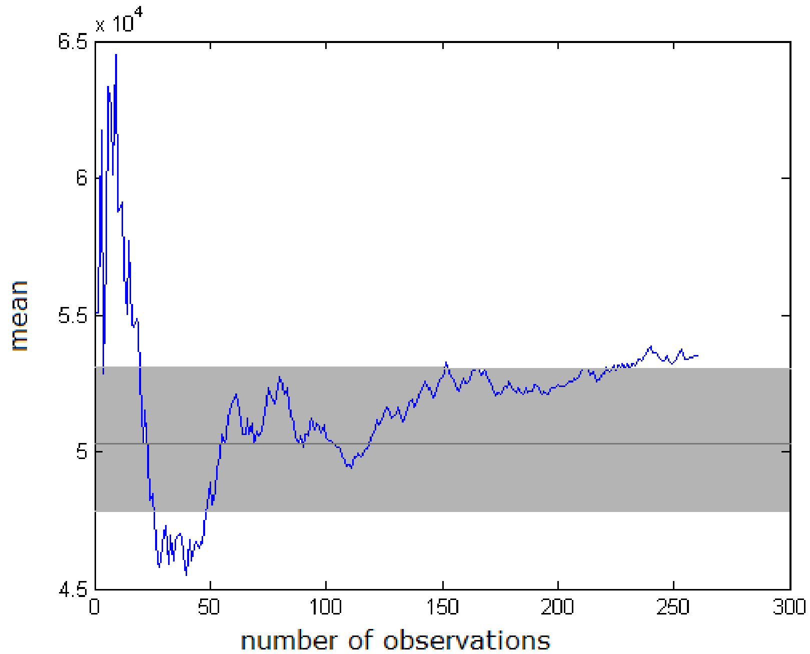

Figure 4.12: Uniform distribution and mean

seen that in the left plot of figure 4.11 (where $d = 0.001$) the average number of observations $k = 170$ for normally distributed sequences. For uniformly distributed sequences the average number of observations is more than 2540.
In statistical theory, a 95% confidence level is often used to evaluate the mean [12]. With the approach given in this thesis, it is difficult to compare the confidence level and the relative error with statistical theory. It is therefore just observed and left as future work to compare the number of necessary observations with statistical theory.
If $d < 0.001$, then more observations are necessary because in a window sometimes the mean of an observation is differing too much from its preceding mean, and more observations must be done, of which the influence is explained above. Summarized, the minimum number of observations for an expected payoff of 0.95 is 170. This result is reached if $d = 0.001$ and $w = 5$.

Experiment 2

In the second experiment, the payoff function is different in order to determine how many observations are needed if costs are involved in the observation of applicants. For a right predicted mean ($q = 0.01$), and the payoff is $1 - c \cdot \frac{v}{n}$, tests are performed for uniformly distributed sequences and situations where $c = 0$, $c = 0.25$, $c = 0.5$, and $c = 1$. In this experiment $d = 0.01$.

Results and discussion

The plots for different values of c are shown in figure 4.13.

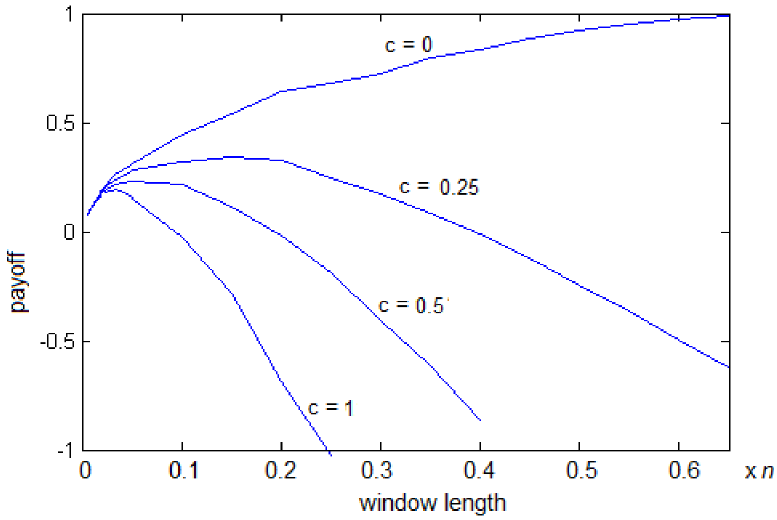

Figure 4.13: Prediction of mean; payoff function with costs

It may be observed that if the costs are increasing, the optimal window length - the window length which obtains the highest payoff - is shorter, and the payoff decreases more rapidly thereafter. Also in this experiment the values are strongly dependent on parameters d and w, payoff variation q and the actual distribution.

Chapter 5: **Conclusions and future work**

In the introduction of this thesis a research goal is defined. In Section 5.1 the findings regarding this goal and the results are discussed. In section 5.2 conclusions are drawn and further studies and recommendations are given.

5.1 Evaluation of the goal

The thesis results in some interesting outcomes for strategies for stopping problems. A simulation environment is created, and for five variants on the secretary problem and one variant to find the mean of a sequence, optimal frame lengths or window lengths are determined, belonging to the maximal expected payoff. For the variant on the secretary problem where a delay is allowed in the rejection and acceptance of applicants, optimal strategies and maximal expected payoffs are found for known situations from literature and for situations which could not be encountered by simple mathematical analysis, and are not known in literature. For cardinal payoffs the optimal frame length is given if sequences are normally distributed, whereas the simulations confirmed the theoretical outcomes for uniformly distributed sequences. Another experiment showed how the optimal frame length is influenced if costs are added to the payoff function of the classical secretary problem.

A payoff function, where not only the acceptance of the best secretary has payoff 1, but also an applicant that has at least threshold value $t(q)$, showed that a small increase in q has an important influence on the optimal frame length. The last variant of the secretary problem, a variant where an applicant may be accepted if it is close to a value that is observed before, shows that there is a strong relation between the allowed deviation with respect to the frame maximum and the allowed deviation with respect to the sequence maximum. It also shows that high payoffs can be obtained in these situations.

Three of the experimental outcomes could be validated by known theoretical outcomes in literature, to confirm that the simulation environment is working correctly.

The five variants on the secretary problem described in this paper are of scientific interest, although it is a fact that there are more interesting variants. Combinations of the concepts introduced in this thesis may lead to many new interesting variants of the secretary problem, which can be examined in the future.

The stopping problem for estimating the mean of a sequence by comparing consecutive means of observed numbers, shows the influence of distributions and costs on the outcomes of the strategy. Many variations on window length w, maxmean variation d, and the exact payoff function show that the strategy is sensitive for the initialization of these parameters. The number of observations necessary for the estimation of the mean for the strategy performed in this thesis could not be compared with known outcomes in statistical literature. It is interesting for future research to compare the minimum number of observations with the statistical sample size.

5.2 General conclusions and future work

Experiments with the variants of stopping problems treated in this thesis lead to a clear insight in possible strategies and the influence of payoff functions to these strategies. The maximization of payoffs for stopping problems based on selecting (one of) the highest value

and estimating the mean value of a sequence, led to interesting results.
The sequences which are input for the strategies in this thesis all are based on the same range. A subject for future research is to study the influence of different ranges of values for sequences and how strategies can react on the ranges if identified.
Because of the many possibilities in variants on strategies and payoff functions, some of which are not covered in this thesis, choices had to be made. There are more interesting variants on the stopping problems than the ones described in this thesis. This thesis is hopefully a basis of some interesting combinations of strategies and payoff functions, which may be experimented and extended in future research.

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