Maastricht University Department of Knowledge Engineering

Studium Generale Lecture Series:

Game Theory

Frank Thuijsman

March 18, 2014

Outline of Series

1. **Introduction to Game Theory**:
   Frank Thuijsman (18-03-2014)
2. **Game Theory and Networks**:
   Tobias Harks (25-03-2014)
3. **Game Theory and Matching**:
   Burak Can (08-04-2014)
4. **Experimental and Behavioral Game Theory**:
   Arno Riedl (15-04-2014)

Outline for Tonight

1. **A Value for 2-Person Zero-Sum Games:**
   John von Neumann (1928)
2. **Equilibria in *n*-Person Games:**
   John Nash (1951)
3. **A Value for *n*-Person Games:**
   Lloyd Shapley (1953)
4. **Understanding a Solution from the Talmud:**
   Bob Aumann and Michael Maschler (1985)

“They’ve led our breakthroughs in game theory.”

2-Person Zero-Sum Games

1	3
4	2

1928

John von Neumann

2-Person Zero-Sum Games

John von Neumann

1 - *p*	**1**	**3**
p	**4**	**2**

2-Person Zero-Sum Games

John von Neumann

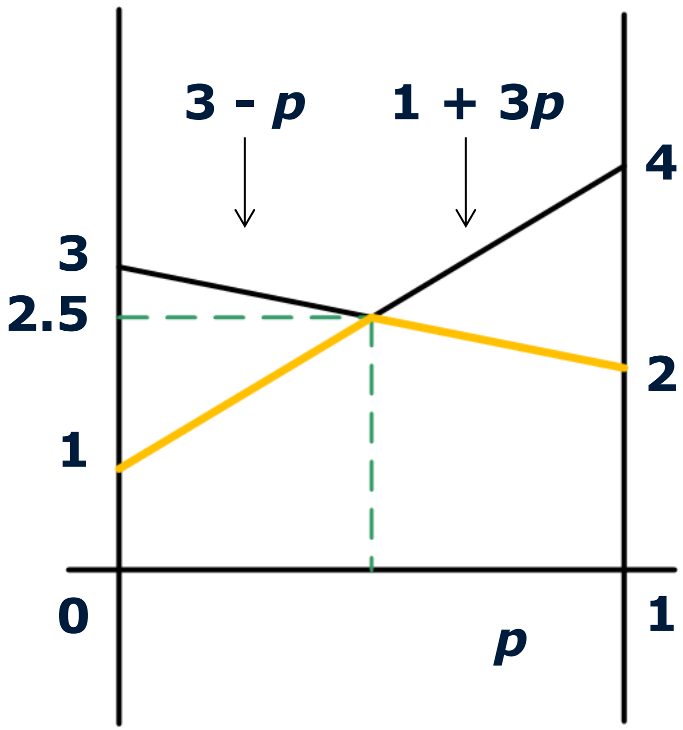
$1 - p$	1	3
p	4	2
	$1 + 3p$	$3 - p$

2-Person Zero-Sum Games

John von Neumann

$1 - p$	1	3
p	4	2
	$1 + 3p$	$3 - p$

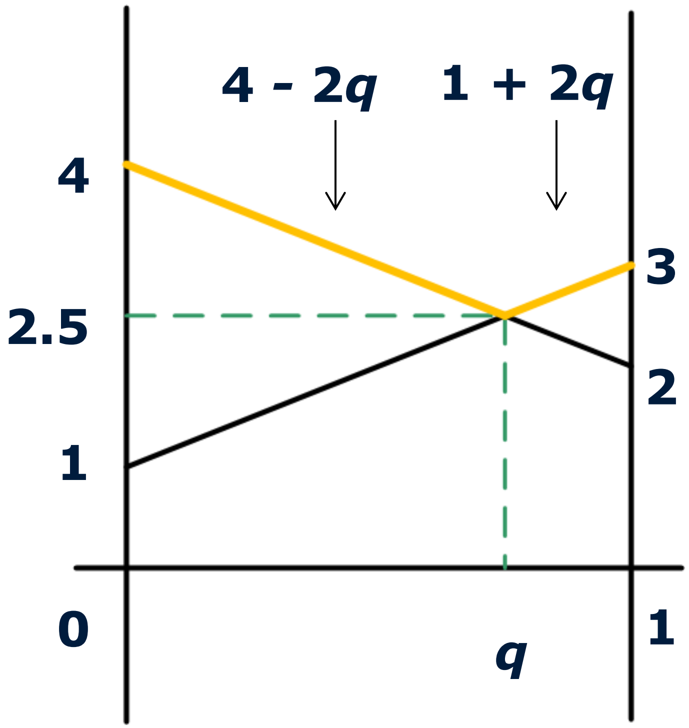
2-Person Zero-Sum Games



1 - *p*	1	3
p	4	2
	1 + 3*p*	3 - *p*

by *p* = 1/2 player 1 maximizes the minimum to get at least 2.5

2-Person Zero-Sum Games



	$1-q$	q	
	1	3	$1+2q$
	4	2	$4-2q$

by $q = 3/4$ player 2 minimizes the maximum to pay at most 2.5

2-Person Zero-Sum Games

John von Neumann

	1/4	3/4
1/2	1	3
1/2	4	2

The number 2.5 is called the *value* of the game.

(1/2,1/2) and (1/4,3/4) are called *optimal strategies* for players 1 and 2.

2-Person Zero-Sum Games

John von Neumann

The MiniMax Theorem

Every finite matrix *A* has a value, i.e.

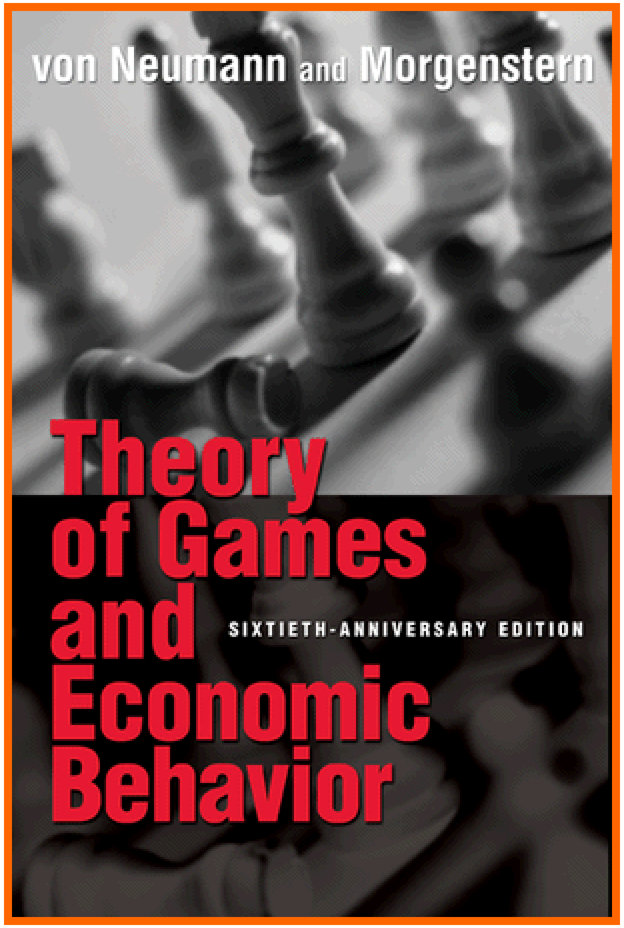
$$\max_x \min_y \; xAy^T \;=\; \min_y \max_x \; xAy^T$$

1928

Launching the Field of Game Theory

John von Neumann

Oskar Morgenstern



1944

From 2-Person Zero-Sum Games …

John von Neumann

1	3
4	2

1928

… to a Solution for *n*-Person Games

1,6	3,-4
4,0	2,5

1951

John von Neumann

John Nash

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… to a Solution for *n*-Person Games

1/2	1,6	3,-4
1/2	4,0	2,5
Player 1 would get	2.5	2.5

John Nash

… to a Solution for *n*-Person Games

1/2	1,6	3,-4
1/2	4,0	2,5
Player 1 would get	2.5	2.5
Player 2 would get	3	0.5

John Nash

... to a Solution for *n*-Person Games

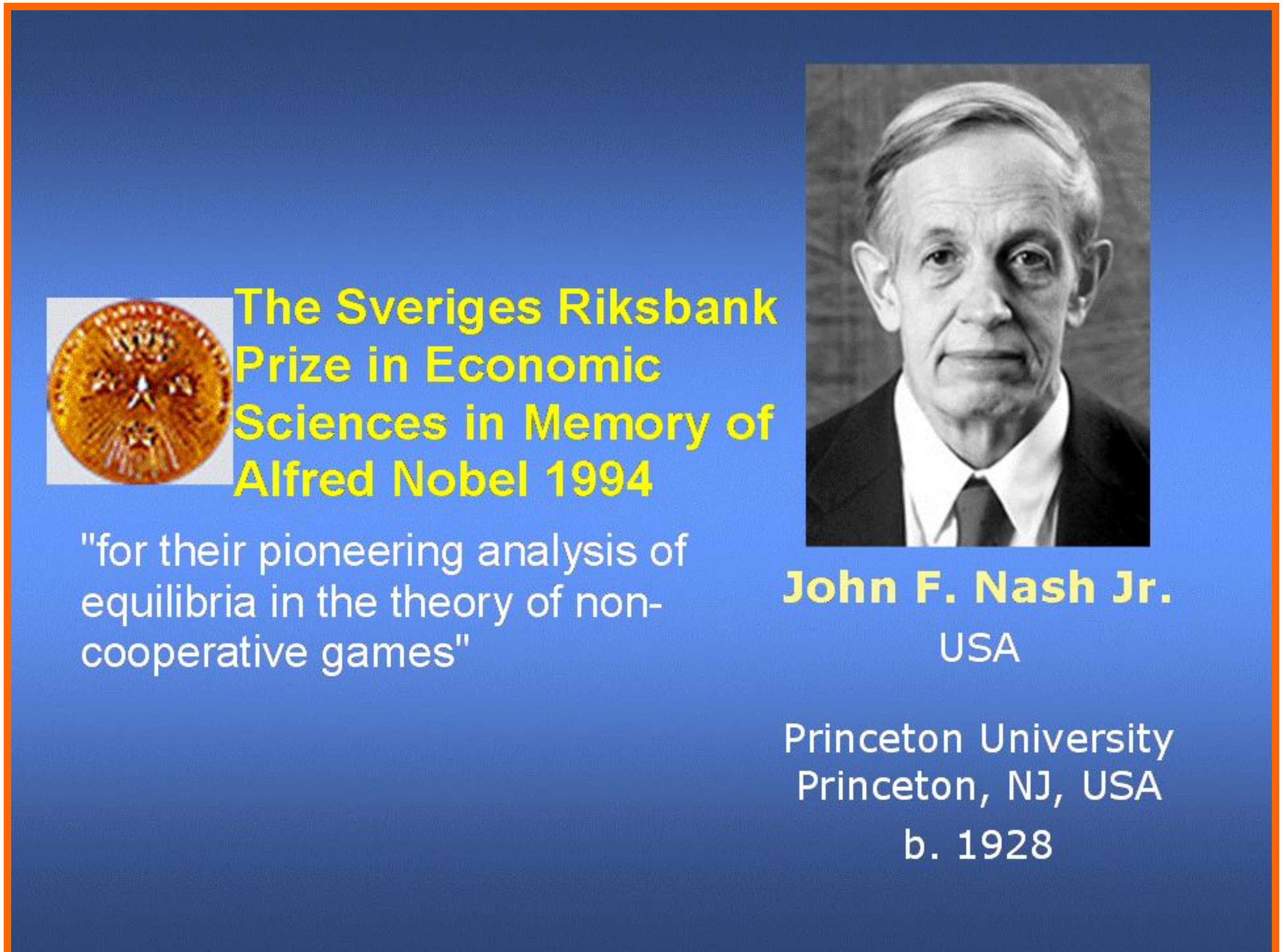
1/2	1,6	3,-4
1/2	4,0	2,5
Player 1 would get	2.5	2.5
Player 2 would get	3	0.5

If *player 1* does what is best for himself, then *player 2* *would prefer to play Left!*

This does NOT give a stable solution!

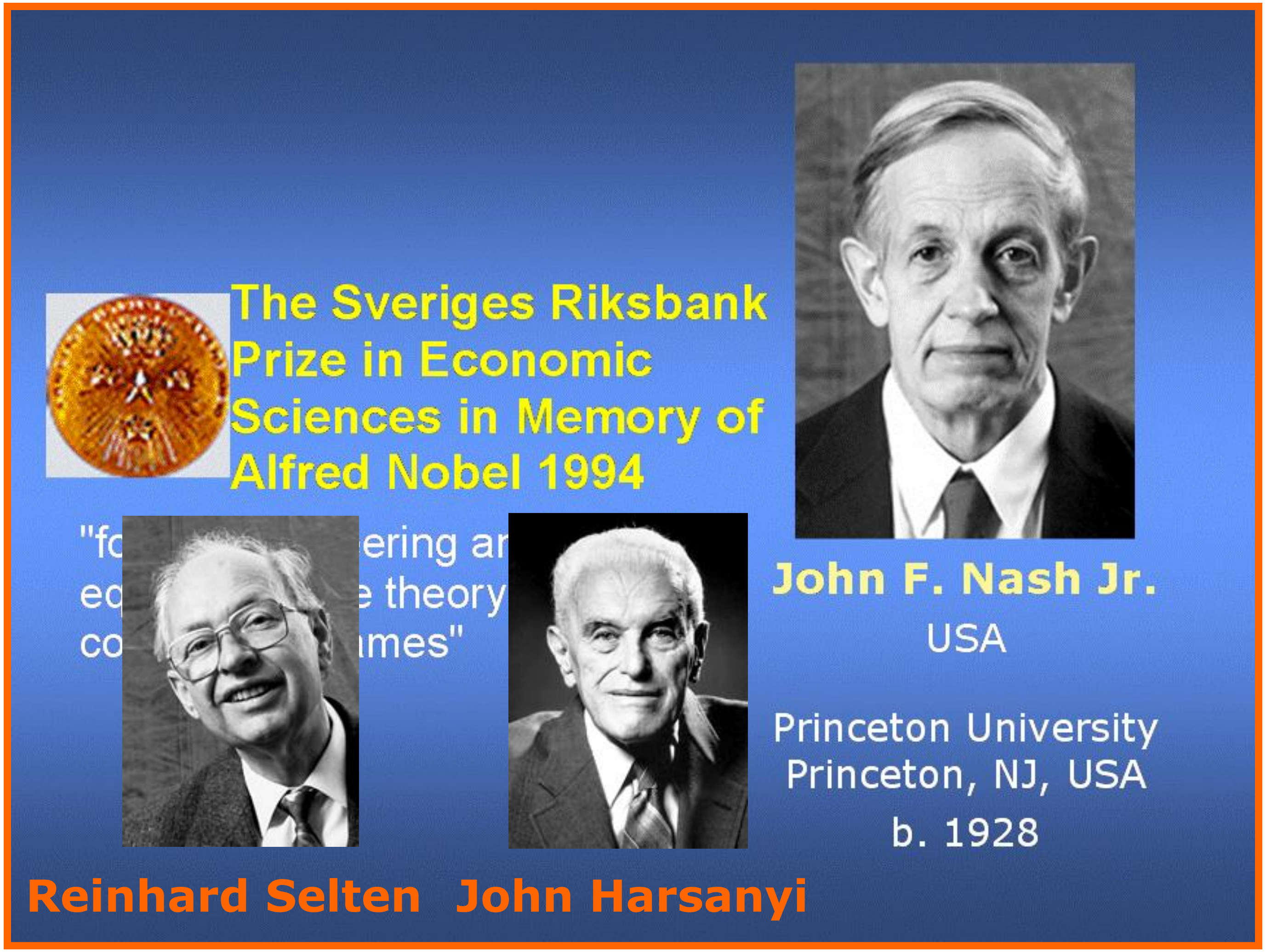
John Nash

Stability in *n*-Player Games

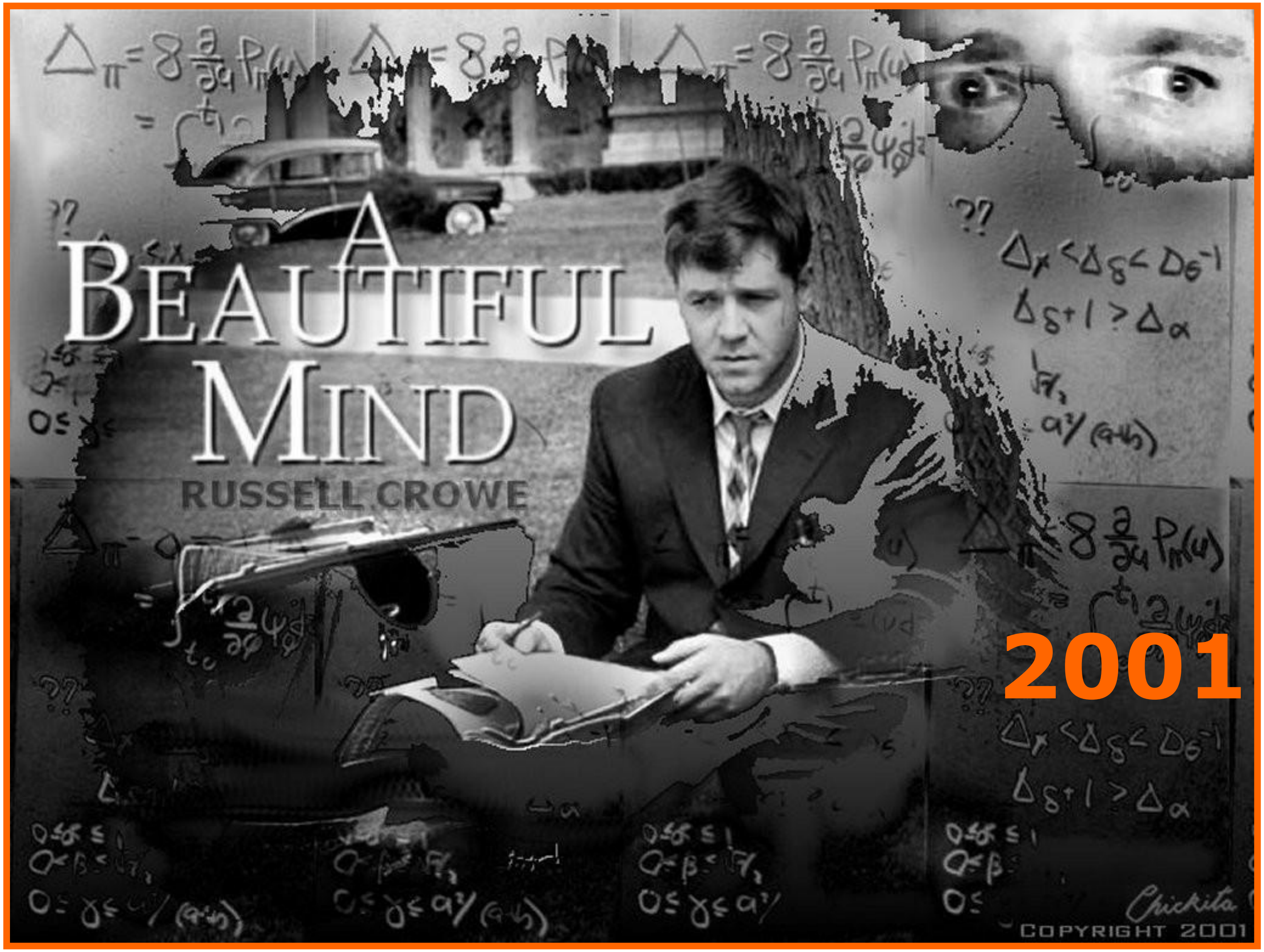


1994

Stability in *n*-Player Games



1994



2001

Play movie

... NOT everyone for himself !!!

1/2	1,6	3,-4
1/2	4,0	2,5
Player 1 would get	2.5	2.5
Player 2 would get	3	0.5

We have to approach it differently!

John Nash

… NOT everyone for himself !!!

$1 - p$	1,6	3,-4
p	4,0	2,5
	$6 - 6p$	$-4 + 9p$

for $p = 2/3$
player 2 is indifferent
between Left and Right;
each gives 2

John Nash

… NOT everyone for himself !!!

	$1 - q$	q
$1 + 2q$	1,6	3,-4
$4 - 2q$	4,0	2,5

for $q = 3/4$
player 1 is indifferent
between Top and Bottom;
each gives 2.5

John Nash

… NOT everyone for himself !!!

	$1 - q$	q
$1 + 2q$	1,6	3,-4
$4 - 2q$	4,0	2,5

((1/3, 2/3), (1/4 , 3/4))
is an *equilibrium* for this game;
player 1 gets 2.5 and
player 2 gets 2.

John Nash

… NOT everyone for himself !!!

An *equilibrium* is a pair of strategies that are best replies to each other.

1951

John Nash

… NOT everyone for himself !!!

Equilibrium Existence:

Every finite *n*-person game has at least one equilibrium.

An *equilibrium* is a pair of strategies that are best replies to each other.

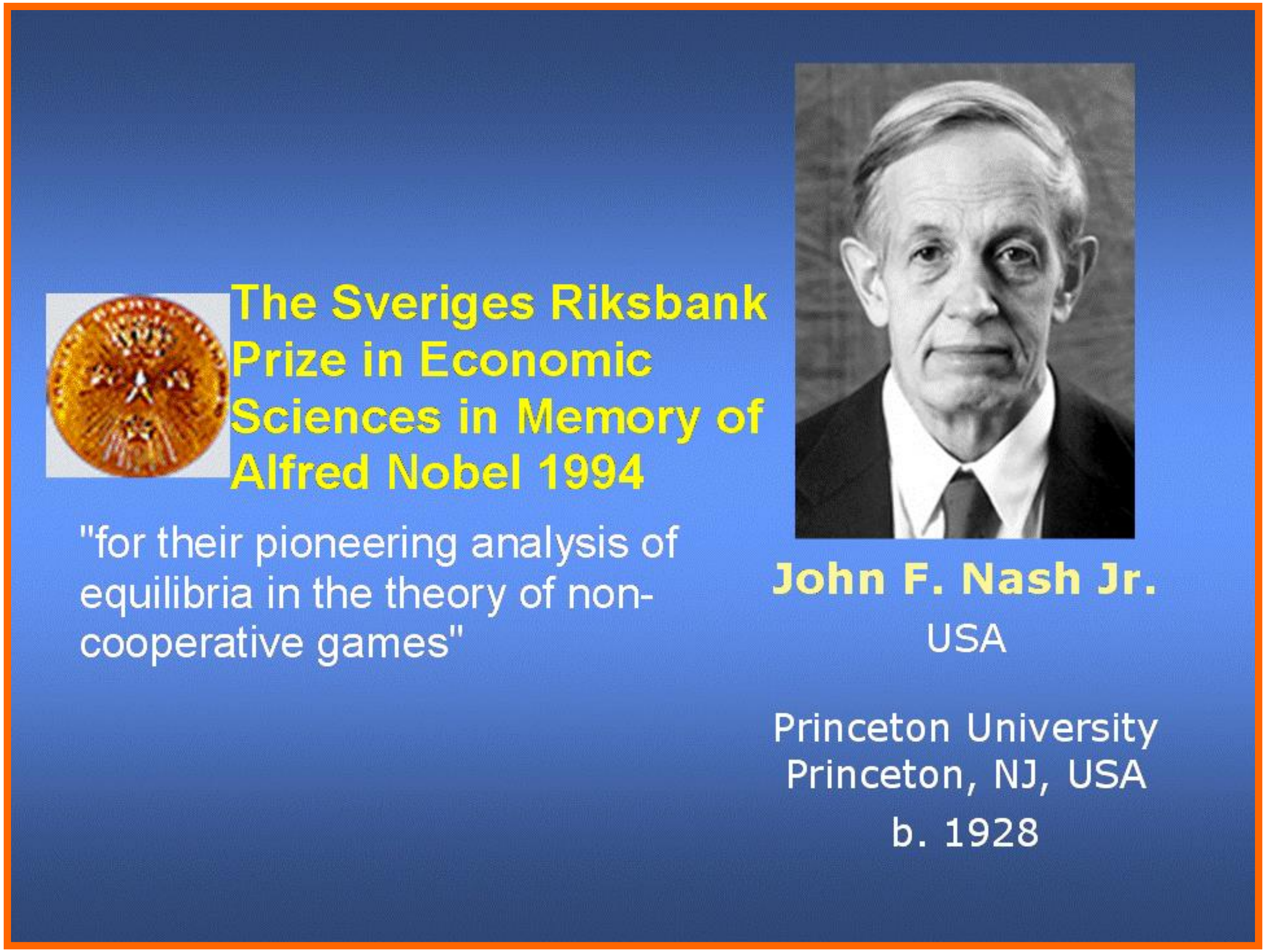
1951

John Nash

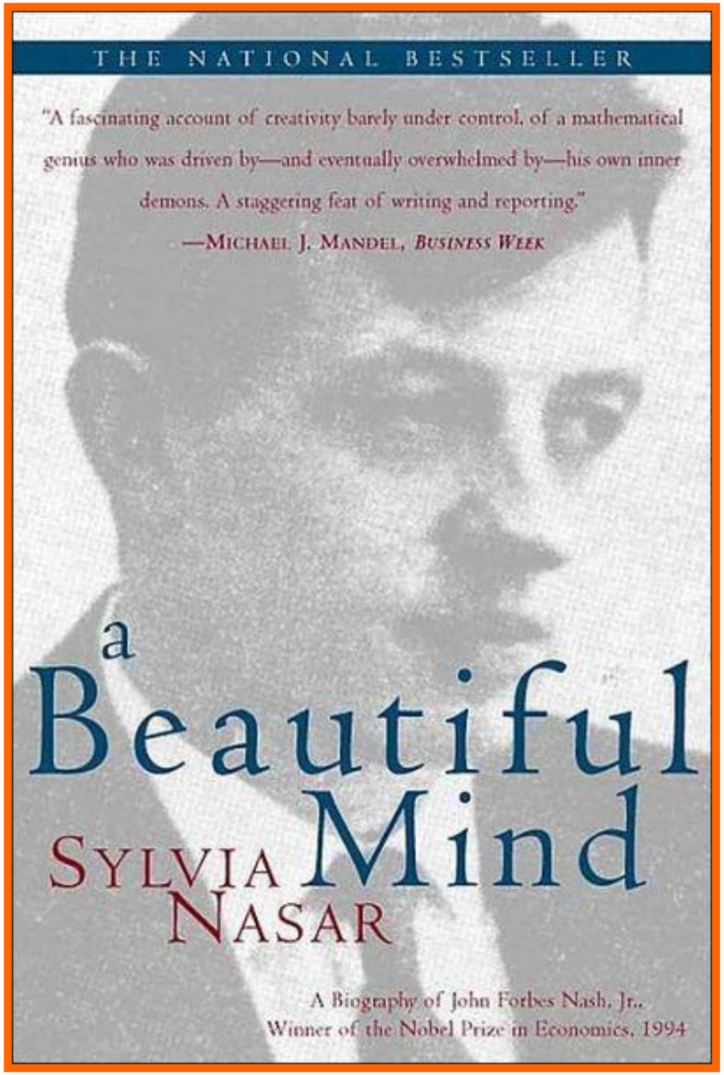
An almost Prisoner's Dilemma

Movie 1
Movie 2
Movie 3

Equilibria in *n*-Player Games



1994



Sylvia Nasar 1998

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Sylvia Nasar 1998

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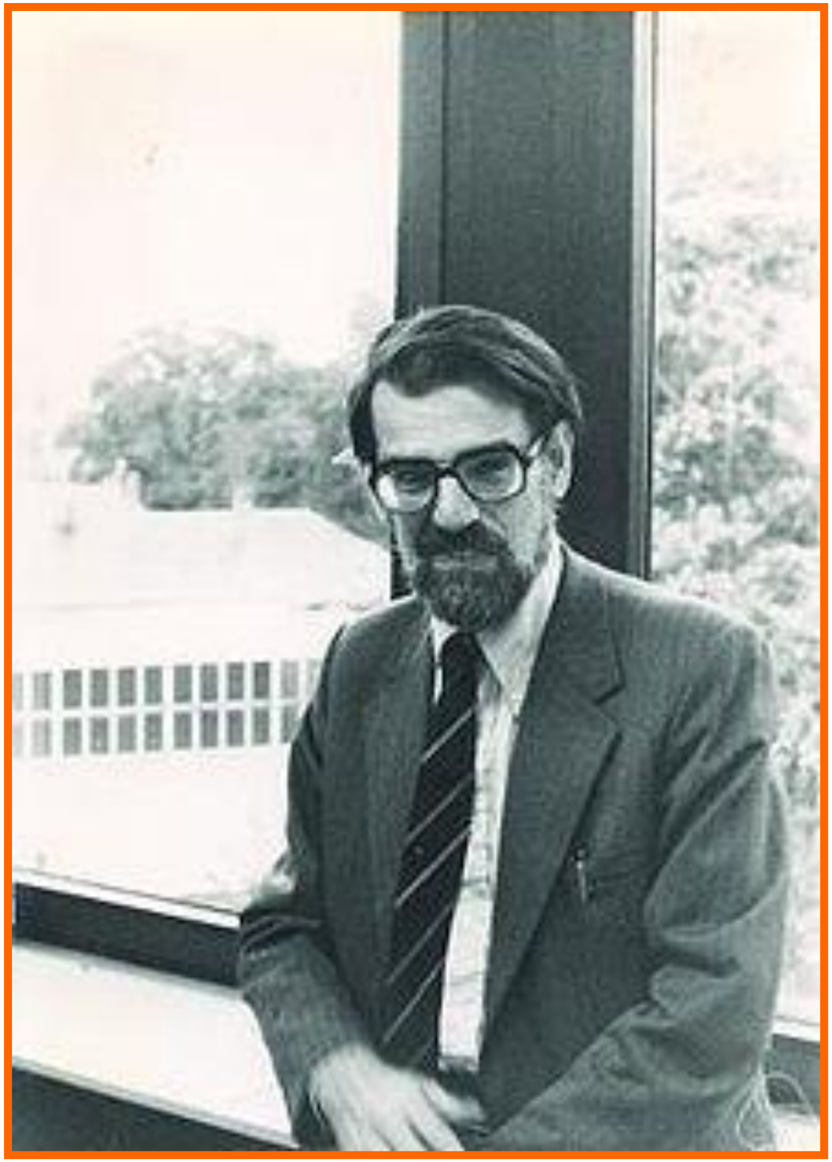


Lloyd Shapley

Quotes from *"A Beautiful Mind"*

[Lloyd Shapley, another pioneer of game theory, described Nash as a graduate student in the late 1940s, when he wrote his seminal papers on game theory: "He was immature, he was obnoxious, he was a brat. What redeemed him was a keen, logical, beautiful mind."

So now you know to whom I owe the title of the biography.]

Quotes from *"A Beautiful Mind"*

[Despite Nash's brilliant dissertation, the consensus at Princeton at the time was that it was Shapley who was the real star of the next generation and inheritor of the von Neumann mantle.

A letter from von Neumann dated January 1954 said: "I know Shapley very well and I think he is VERY good."]

Quotes from *"A Beautiful Mind"*

[Nash was one of the first students Shapley met at the Graduate College. For a time, they shared a bathroom. Both of them attended Tucker's game theory seminar every Thursday, now run by Kuhn and Gale while Tucker was at Stanford. The best way to describe the impression Nash made on Shapley when the two first talked about mathematics is to say that Nash took Shapley's breath away. Shapley could, of course, see what the others saw — the childishness, brattiness, obnoxiousness — but he saw a great deal more. He was dazzled by what he would later describe as Nash's "keen, beautiful, logical mind."]

Quotes from *"A Beautiful Mind"*

[As for Nash, starved for affection, how could he not be drawn to Shapley? In Nash's eyes, Shapley had it all. A brilliant mathematician. War hero. Harvard man. A son of Harlow. Favorite of von Neumann and, soon, of Tucker as well. Shapley, who was popular with faculty and students alike, was one of the very few around Princeton, other than Milnor, who could really hold Nash's attention in a mathematical conversation, challenge him, and help him to pursue the implications of his own reasoning. And, for that reason — along with his open admiration and obvious sympathy — he was one who could engage Nash's emotions.]

Lloyd Shapley

2012

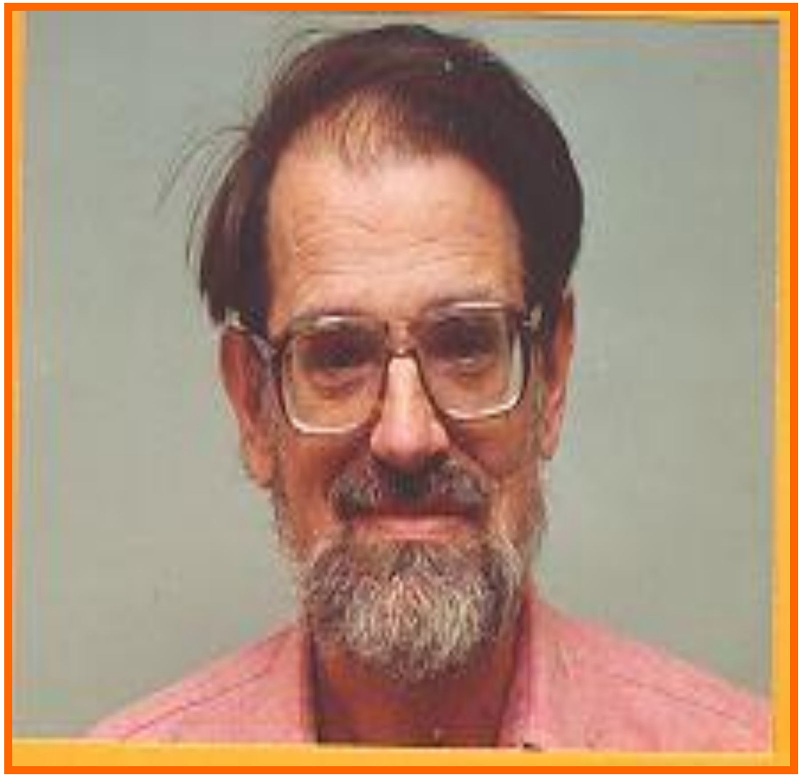
Alvin Roth and Lloyd Shapley

Lloyd Shapley

Marriage Problems

College admissions and the stability of marriage,
***American Mathematical Monthly* 69, 1962**

David Gale



Lloyd S. Shapley

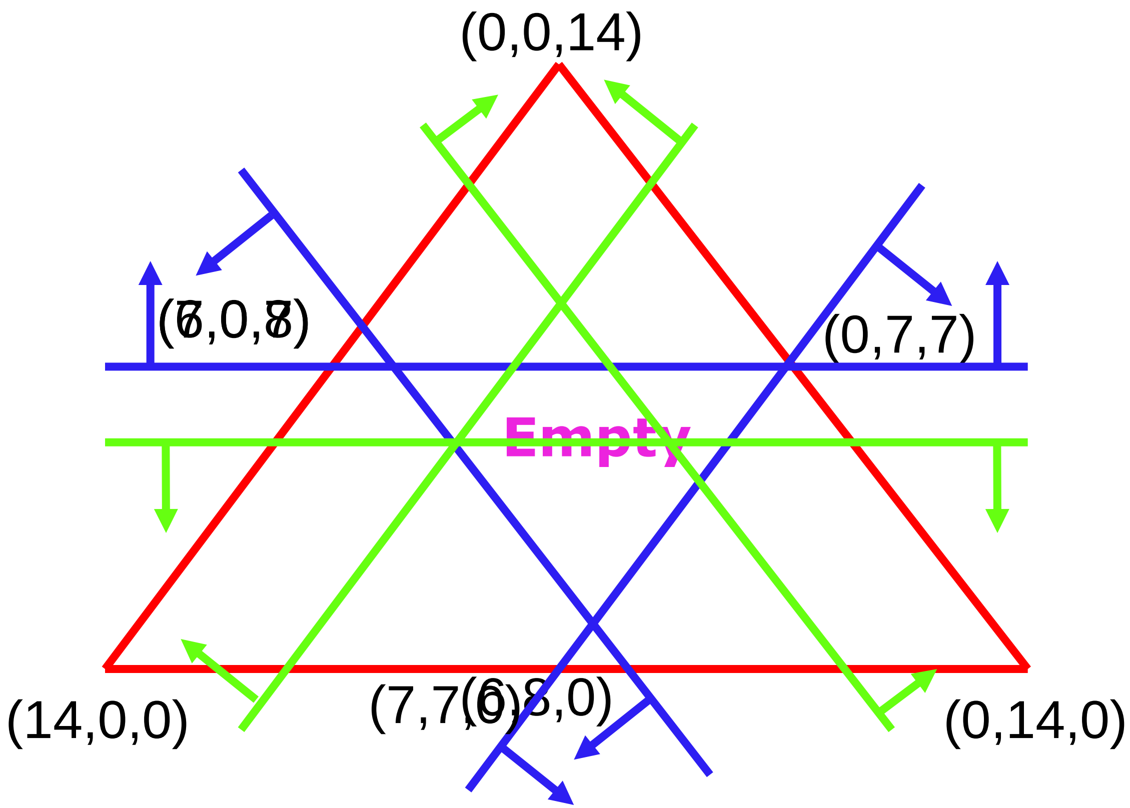
Cooperative Games

S	Ø	A	B	C	AB	AC	BC	ABC
v(S)	0	6	7	7	9	11	11	14

Sharing costs or profits in a fair way that corresponds to the values of each of the coalitions

The Core

S	Ø	A	B	C	AB	AC	BC	ABC
v(S)	0	6	7	7	9	11	11	14



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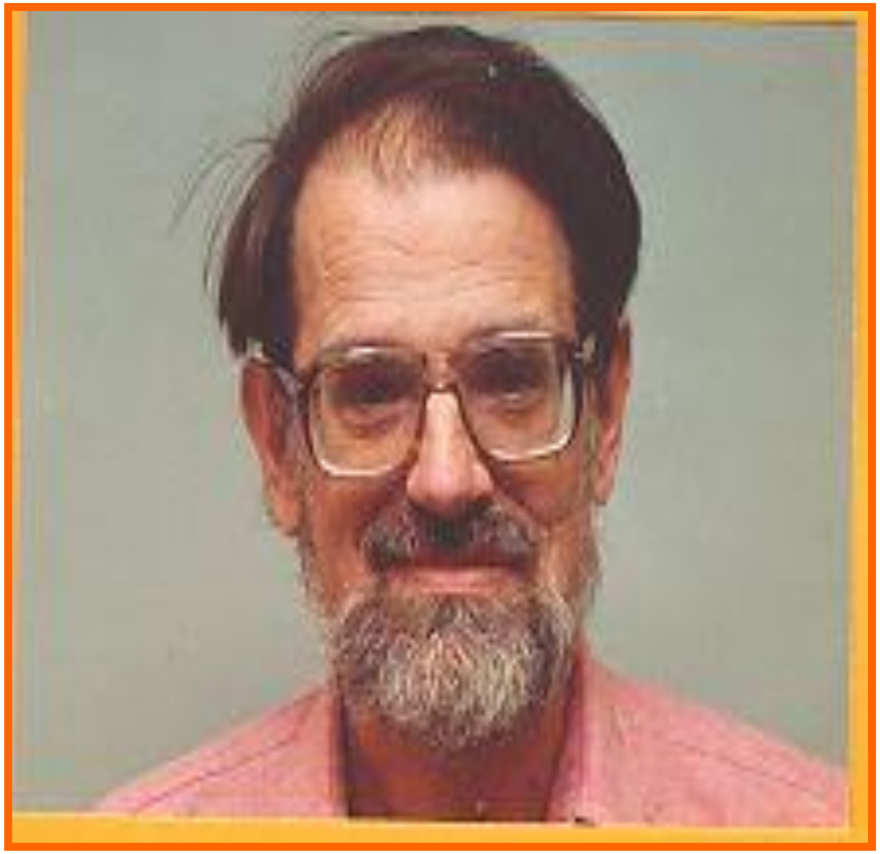

Lloyd S. Shapley

A value for *n*-person games, In: *Contributions to the Theory of Games,* Kuhn and Tucker (eds), Princeton, 1953

A *Value* for *n*-Player Games

For coopeartive games there is only one solution principle that satisfies the properties of:

- Anonimity

- Efficiency

- Dummy

- Additivity

The Shapley-value Φ gives every player the average of his marginal contributions.

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The Shapley-value

S	Ø	A	B	C	AB	AC	BC	ABC
v(S)	0	6	7	7	9	11	11	14

Marginal contributions

	A	**B**	**C**
A-B-C	6	3	5
A-C-B	6	3	5
B-A-C	2	7	5
B-C-A	3	7	4
C-A-B	4	3	7
C-B-A	3	4	7
Sum:	24	27	33
Φ:	**4**	**4.5**	**5.5**

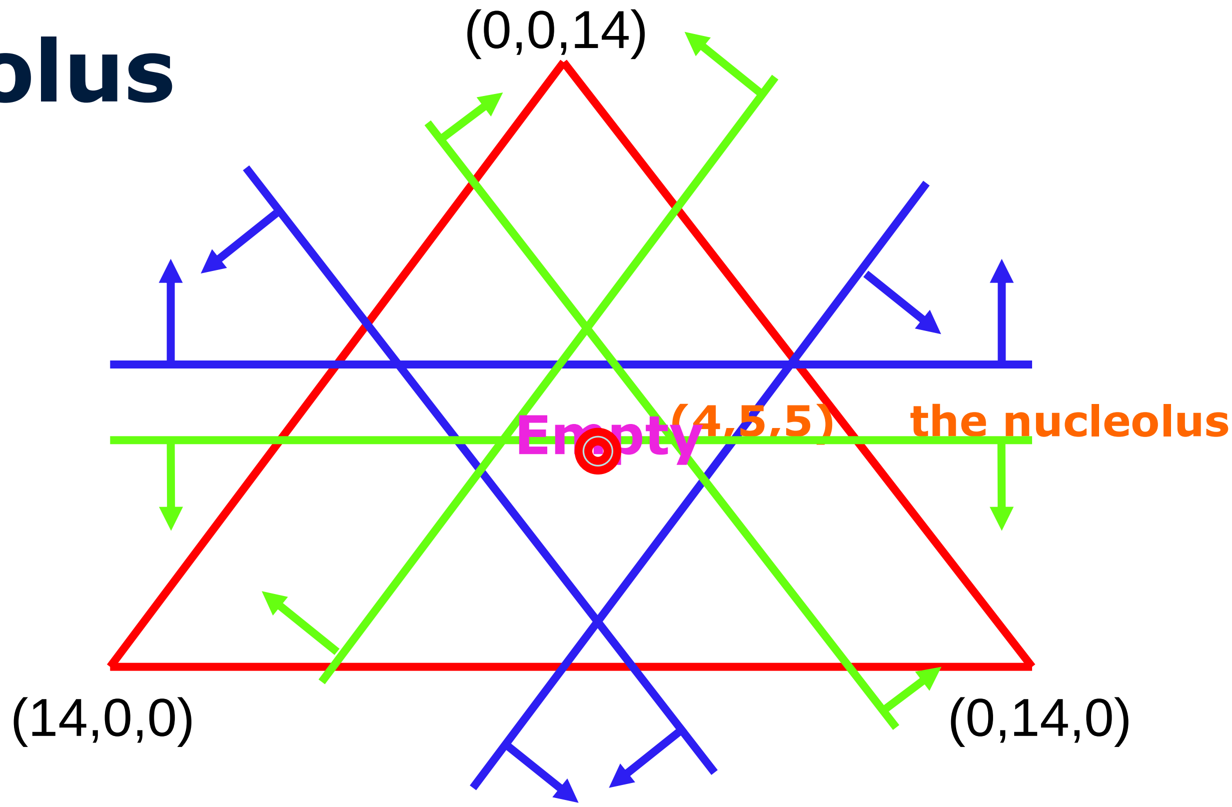
David Schmeidler

The nucleolus of a characteristic function game,
***SIAM Journal of Applied Mathematics* 17, 1969**

S	Ø	A	B	C	AB	AC	BC	ABC
v(S)	0	6-x	7-x	7-x	9-x	11-x	11-x	14

The Nucleolus

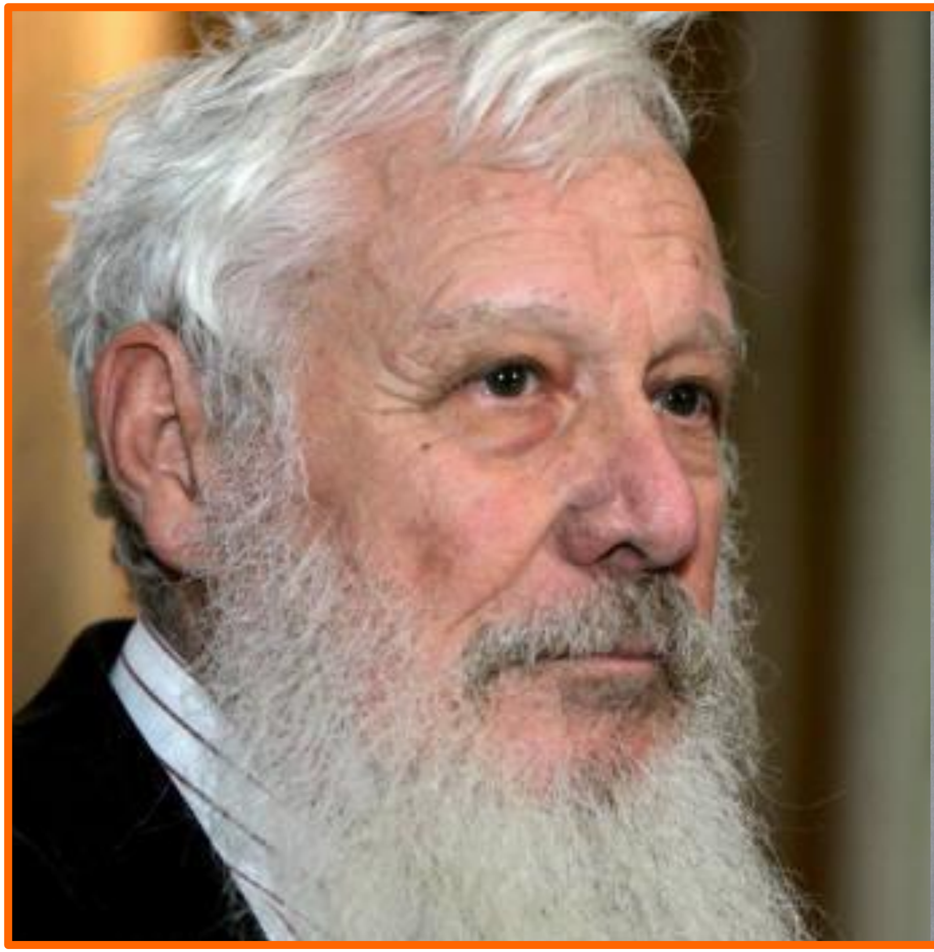
Φ = (4, 4.5, 5.5)



2005 Nobel Prize

Thomas Schelling **Bob Aumann**

Game theoretic analysis of a bankruptcy problem from the Talmud, *Journal of Economic Theory* 36, 1985



Bob Aumann

Michael Maschler

A problem of rights arbitration from the Talmud, *Mathematical Social Sciences* 2, 1982

Barry O'Neill

Three Widows

A Bankruptcy Problem from the Talmud

Kethuboth, Fol. 93a, Babylonian Talmud, Epstein, ed, 1935

"If a man who was married to three wives died and the kethubah of one was 100 zuz, of the other 200 zuz, and of the third 300 zuz, and the estate was worth only 100 zuz, then the sum is divided equally.

If the estate was worth 200 zuz then the claimant of the 100 zuz receives 50 zuz and the claimants respectively of the 200 and the 300 zuz receive each 75 zuz.

If the estate was worth 300 zuz then the claimant of the 100 zuz receives 50 zuz and the claimant of the 200 zuz receives 100 zuz while the claimant of the 300 zuz receives 150 zuz.

Similarly if three persons contributed to a joint fund and they had made a loss or a profit then they share in the same manner."

A Bankruptcy Problem from the Talmud

Estate

Widow

	100	200	300
100	33.33		
200	33.33		
300	33.33		

A Bankruptcy Problem from the Talmud

Kethuboth, Fol. 93a, Babylonian Talmud, Epstein, ed, 1935

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A Bankruptcy Problem from the Talmud

Estate

Widow

	100	200	300
100	33.33	50	
200	33.33	75	
300	33.33	75	

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A Bankruptcy Problem from the Talmud

Kethuboth, Fol. 93a, Babylonian Talmud, Epstein, ed, 1935

“If a man who was married to three wives died and the kethubah of one was 100 zuz, of the other 200 zuz, and of the third 300 zuz, and the estate was worth only 100 zuz, then the sum is divided equally.

If the estate was worth 200 zuz then the claimant of the 100 zuz receives 50 zuz and the claimants respectively of the 200 and the 300 zuz receive each 75 zuz.

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A Bankruptcy Problem from the Talmud

Estate

Widow		100	200	300
	100	33.33	50	50
	200	33.33	75	100
	300	33.33	75	150

A Bankruptcy Problem from the Talmud

Estate

Widow		100	200	300
	100	33.33	50	50
	200	33.33	75	100
	300	33.33	75	150

Equal **???** **Proportional**

A Bankruptcy Problem from the Talmud

Kethuboth, Fol. 93a, Babylonian Talmud, Epstein, ed, 1935

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A Bankruptcy Problem from the Talmud

Estate

Widow		100	200	300
	100	33.33	50	50
	200	33.33	75	100
	300	33.33	75	150

Equal **???** **Proportional**

"Other amounts shared **in the same way**." ?????

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A Bankruptcy Problem from the Talmud

Estate

Widow		100	200	300
	100	33.33	50	50
	200	33.33	75	100
	300	33.33	75	150

Equal **???** **Proportional**

"Other amounts shared **in the same way**." ?????

How to share 400?

What if a fourth widow claims 400?

The Talmud Problem and Cooperative Games

	100	200	300
100			
200			
300			

The value of any coalition S is the amount that remains, when we first pay the claims of the other players.

S	Ø	A	B	C	AB	AC	BC	ABC
v(S)	0	0	0	0	0	0	100	200

The Talmud Problem and Cooperative Games

	100	200	300
A 100			
B 200			
C 300			

The value of any coalition S is the amount that remains, when we first pay the claims of the other players.

S	Ø	A	B	C	AB	AC	BC	ABC
v(S)	0	0	0	0	0	0	0	100

The Talmud Problem and Cooperative Games

	100	200	300
100			
200			
300			

The value of any coalition S is the amount that remains, when we first pay the claims of the other players.

S	Ø	A	B	C	AB	AC	BC	ABC
v(S)	0	0	0	0	0	100	200	300

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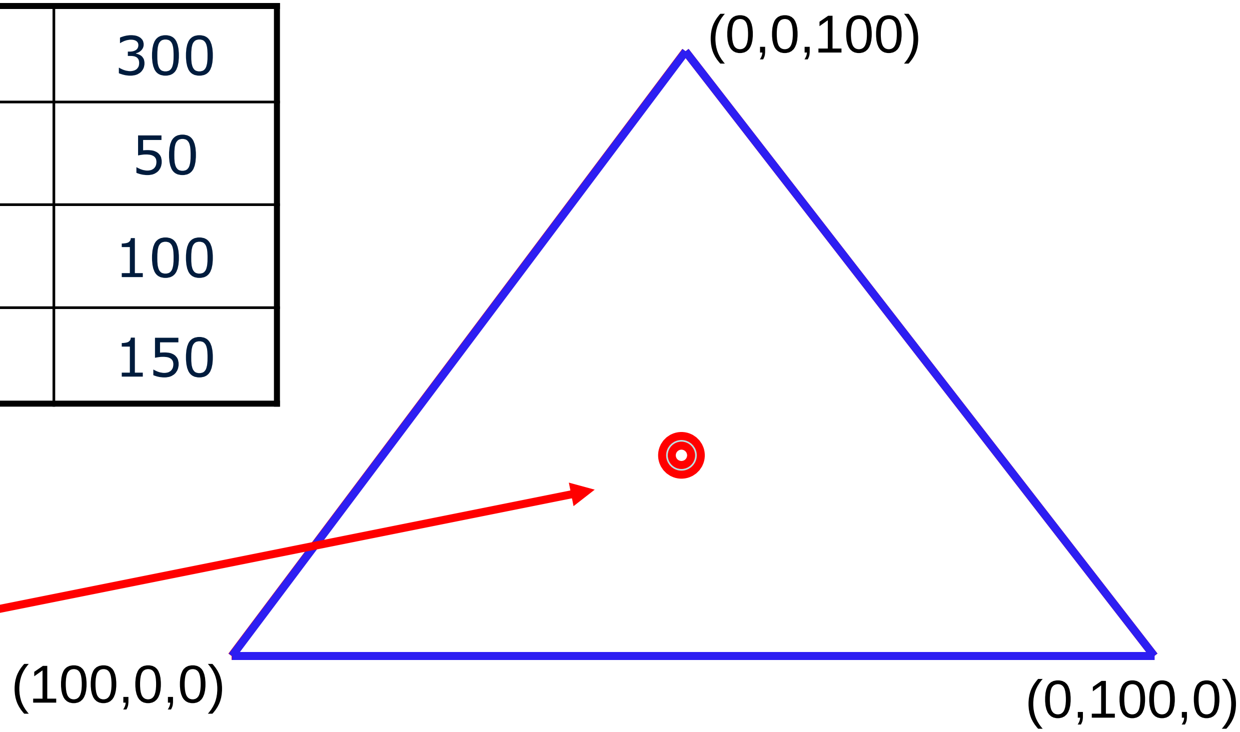
Talmud-games

	100	200	300
A 100	33.33	50	50
B 200	33.33	75	100
C 300	33.33	75	150

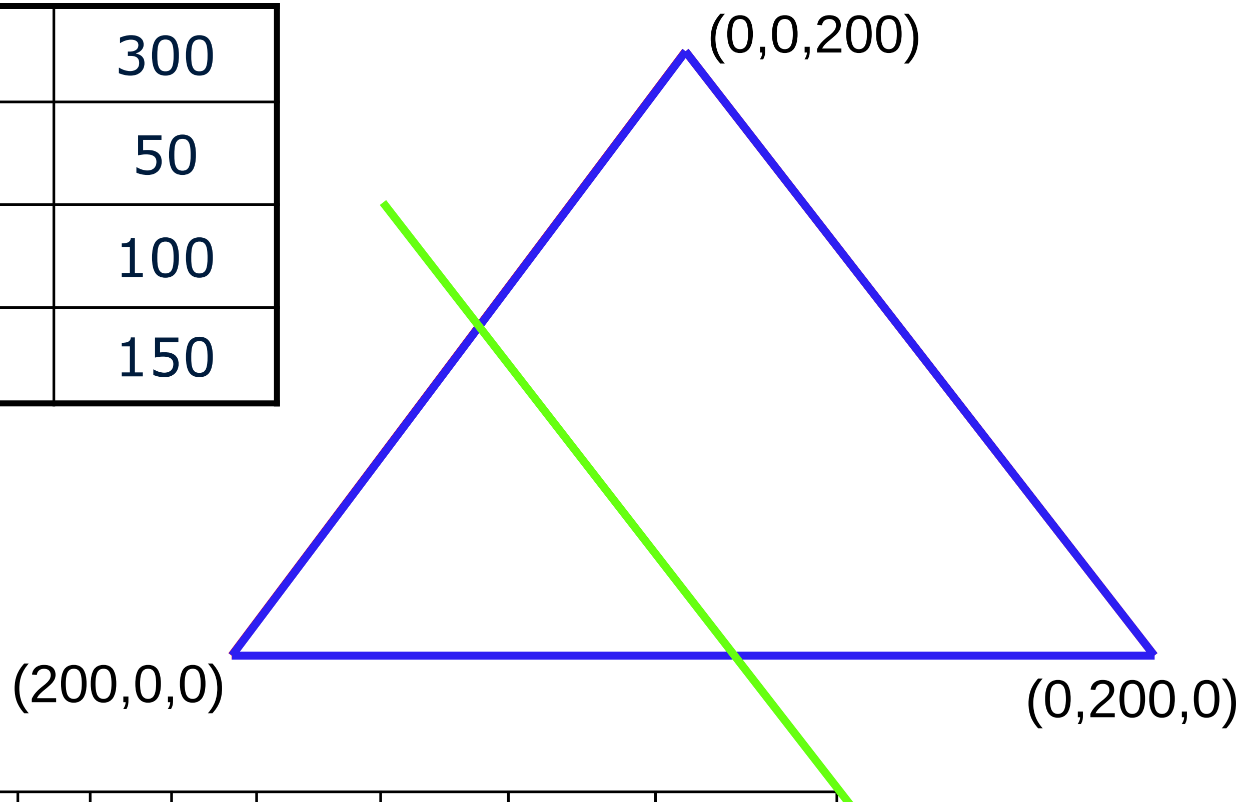
the nucleolus



S	Ø	A	B	C	AB	AC	BC	ABC
v(S)	0	0	0	0	0	0	0	100

Talmud-games

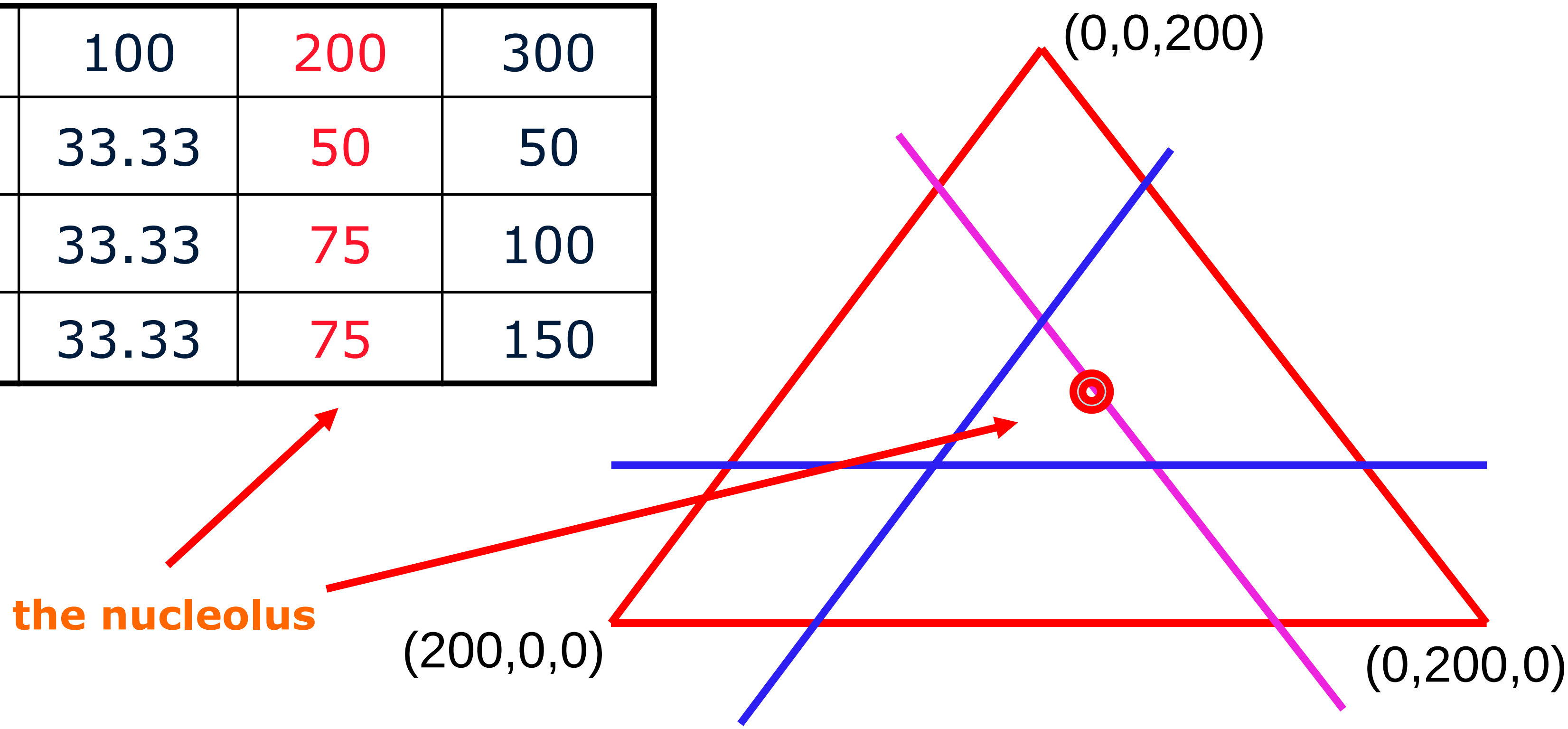
	100	200	300
100	33.33	50	50
200	33.33	75	100
300	33.33	75	150



S	Ø	A	B	C	AB	AC	BC	ABC
v(S)	0	0	0	0	0	0	100	200

Talmud-games

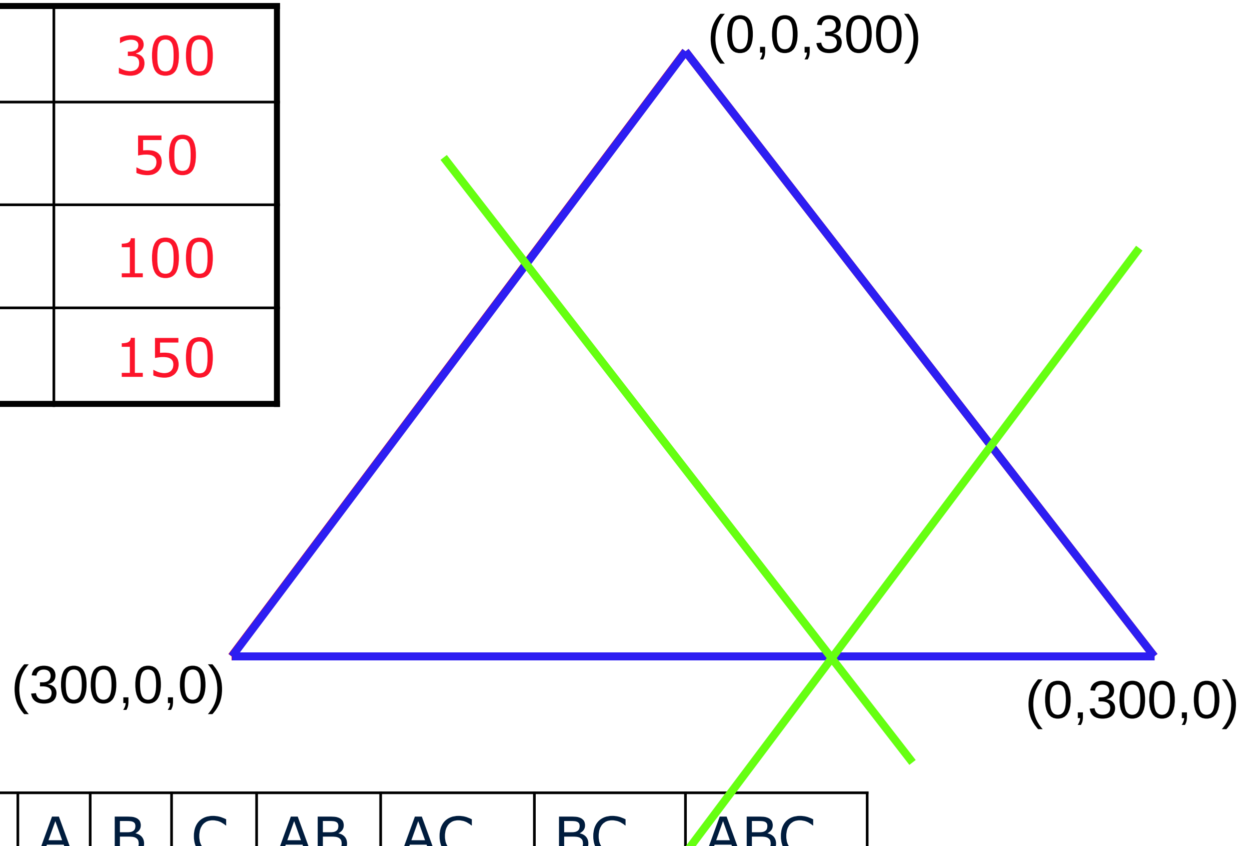
	100	200	300
100	33.33	50	50
200	33.33	75	100
300	33.33	75	150



S	Ø	A	B	C	AB	AC	BC	ABC
v(S)	0	0	0	0	0	0	100	200

Talmud-games

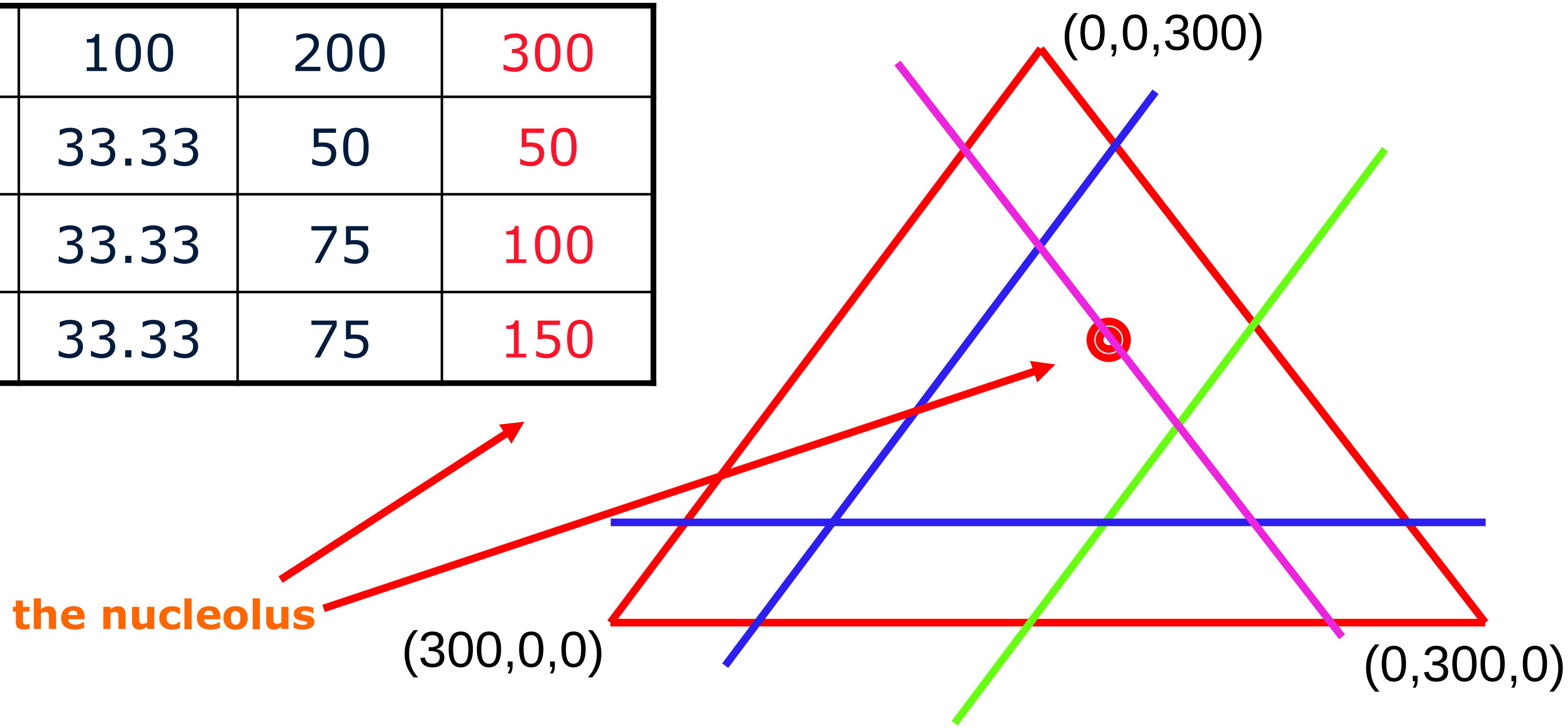
	100	200	300
100	33.33	50	50
200	33.33	75	100
300	33.33	75	150



S	Ø	A	B	C	AB	AC	BC	ABC
v(S)	0	0	0	0	0	100	200	300

Talmud-games

	100	200	300
100	33.33	50	50
200	33.33	75	100
300	33.33	75	150



S	Ø	A	B	C	AB	AC	BC	ABC
v(S)	0	0	0	0	0	100	200	300

A Bankruptcy Problem from the Talmud

Estate

		100	200	300
Widow	100	33.33	50	50
	200	33.33	75	100
	300	33.33	75	150

Equal **???** **Proportional**

"Other amounts shared **in the same way**." ?????

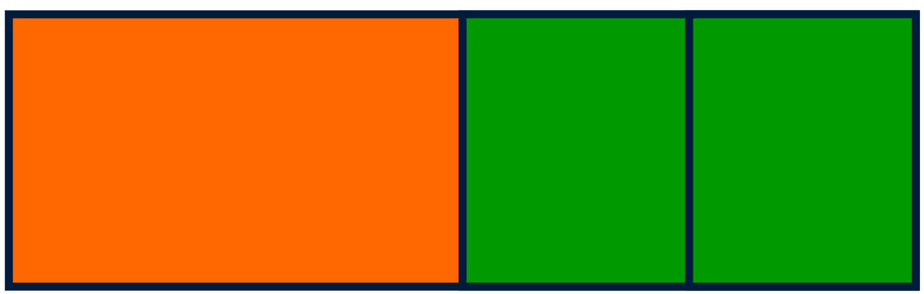
How to share 400?

Just calculate the nucleolus !!!!!

The Answer

Baba Metzia 2a, Fol. 1, Babylonian Talmud, Epstein, ed, 1935

“Two hold a garment;
one claims it all, the other claims half.
Then one gets 3/4 , while the other gets 1/4.”



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Consistency

Estate

Widow		100	200	300
	100	33.33	50	50
Widow	200	33.33	75	100
	300	33.33	75	150

A closer look at these numbers

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Consistency

Estate

Widow

	100	200	300
100	33.33	50	50
200	33.33	75	100
300	33.33	75	150

Consistency

Estate

Widow

	100	200	300
100	33.33	50	50
200	33.33	75	100
300	33.33	75	150

Consistency

Estate

Widow

	100	200	300
100	33.33	50	50
200	33.33	75	100
300	33.33	75	150

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Consistency

Estate

Widow

		100	200	300
Widow	100	33.33	50	50
	200	33.33	75	100
	300	33.33	75	150

Consistency

Estate

		100	200	300
		100	200	300
Widow	100	$\boxed{33.33}$	50	50
	200	33.33	75	100
	300	$\boxed{33.33}$	75	150

The questions remain:

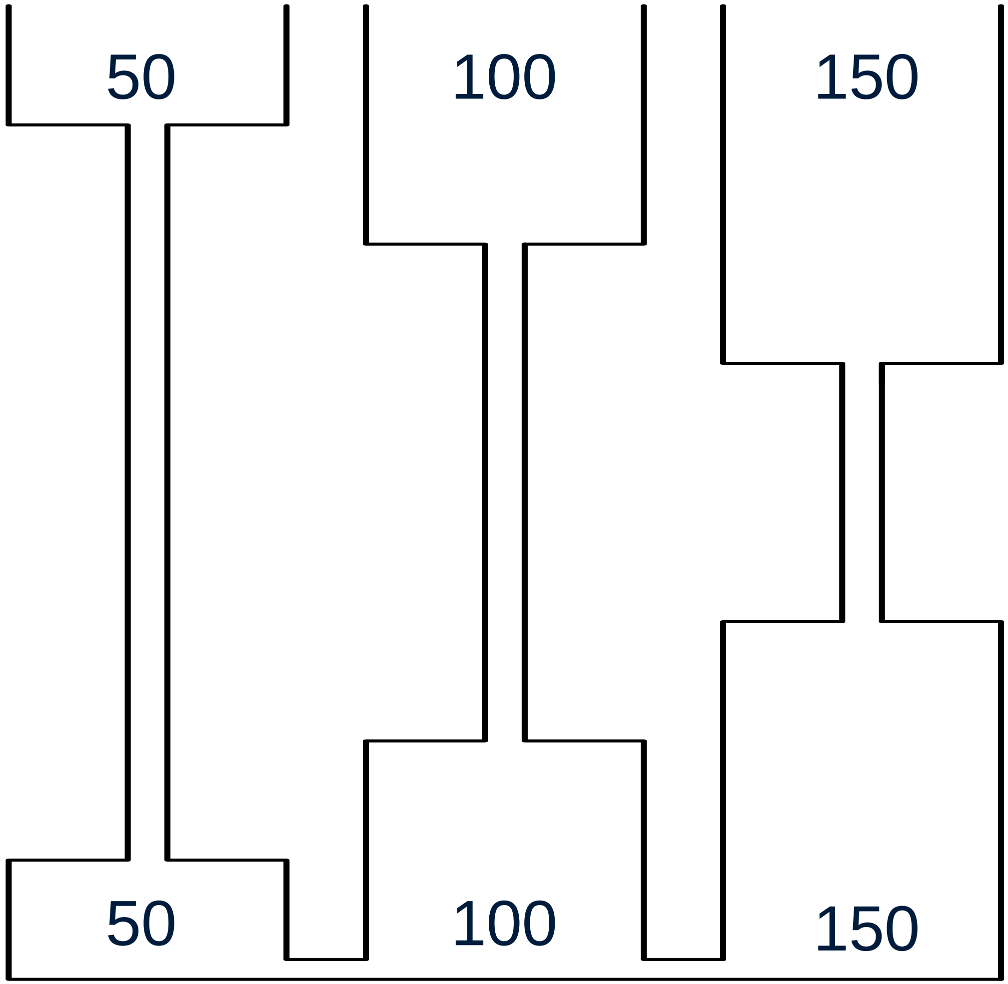
How to share 400?

What if a fourth widow claims 400?

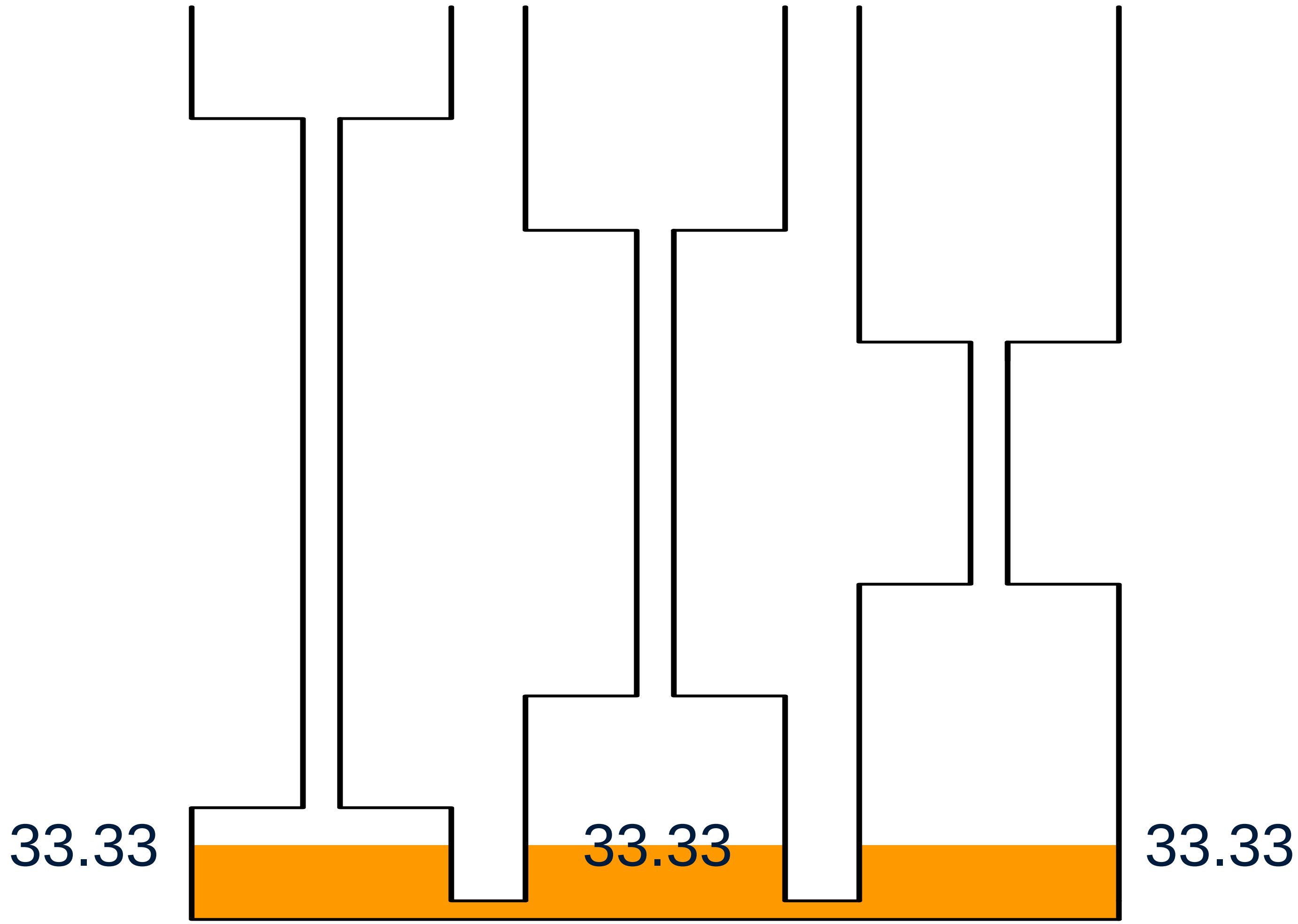
‘Hydraulic’ rationing, *Mathematical Social Sciences* 40, 2000

Marek M. Kaminski

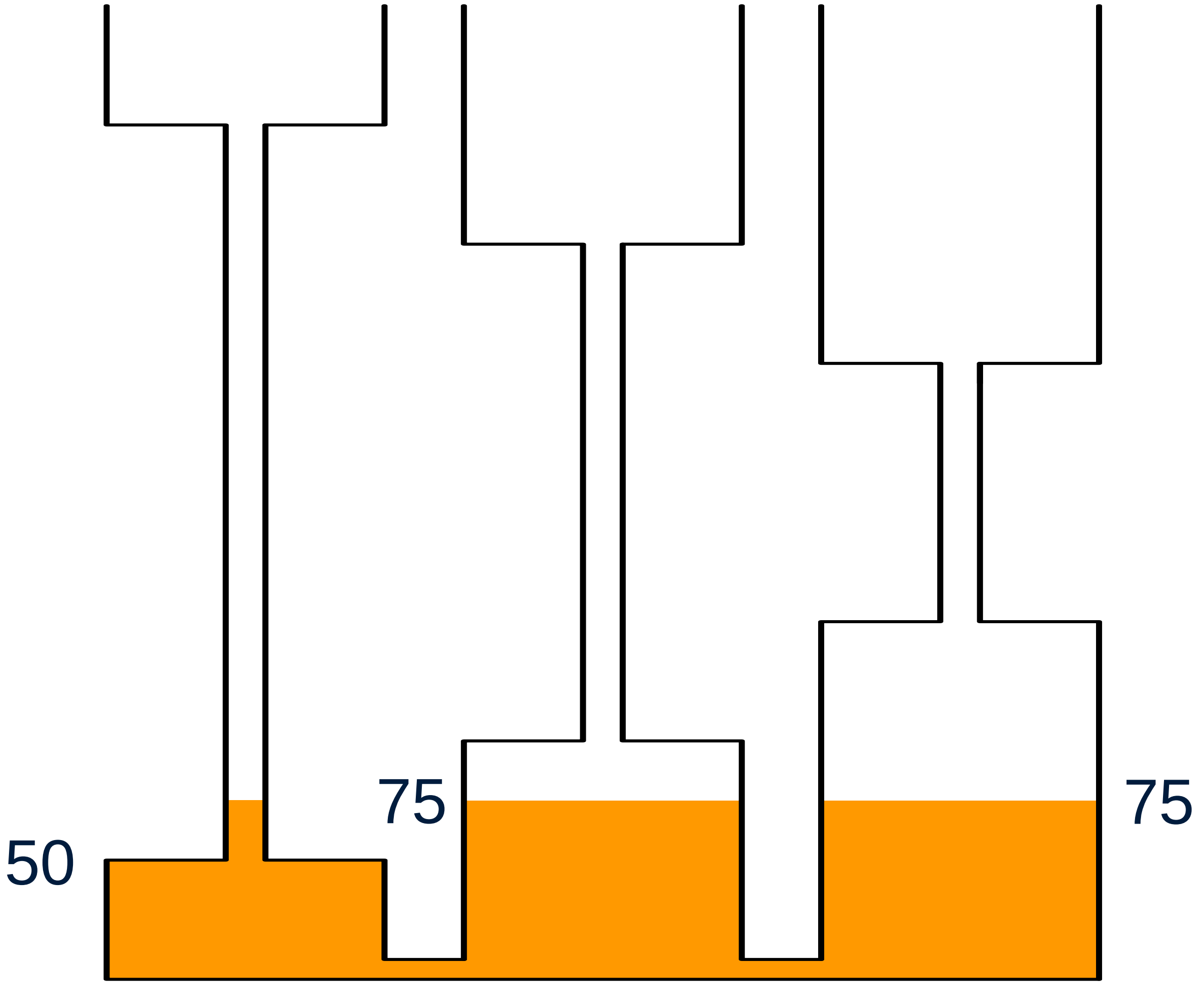
Communicating Vessels



Communicating Vessels: 100



Communicating Vessels: 200



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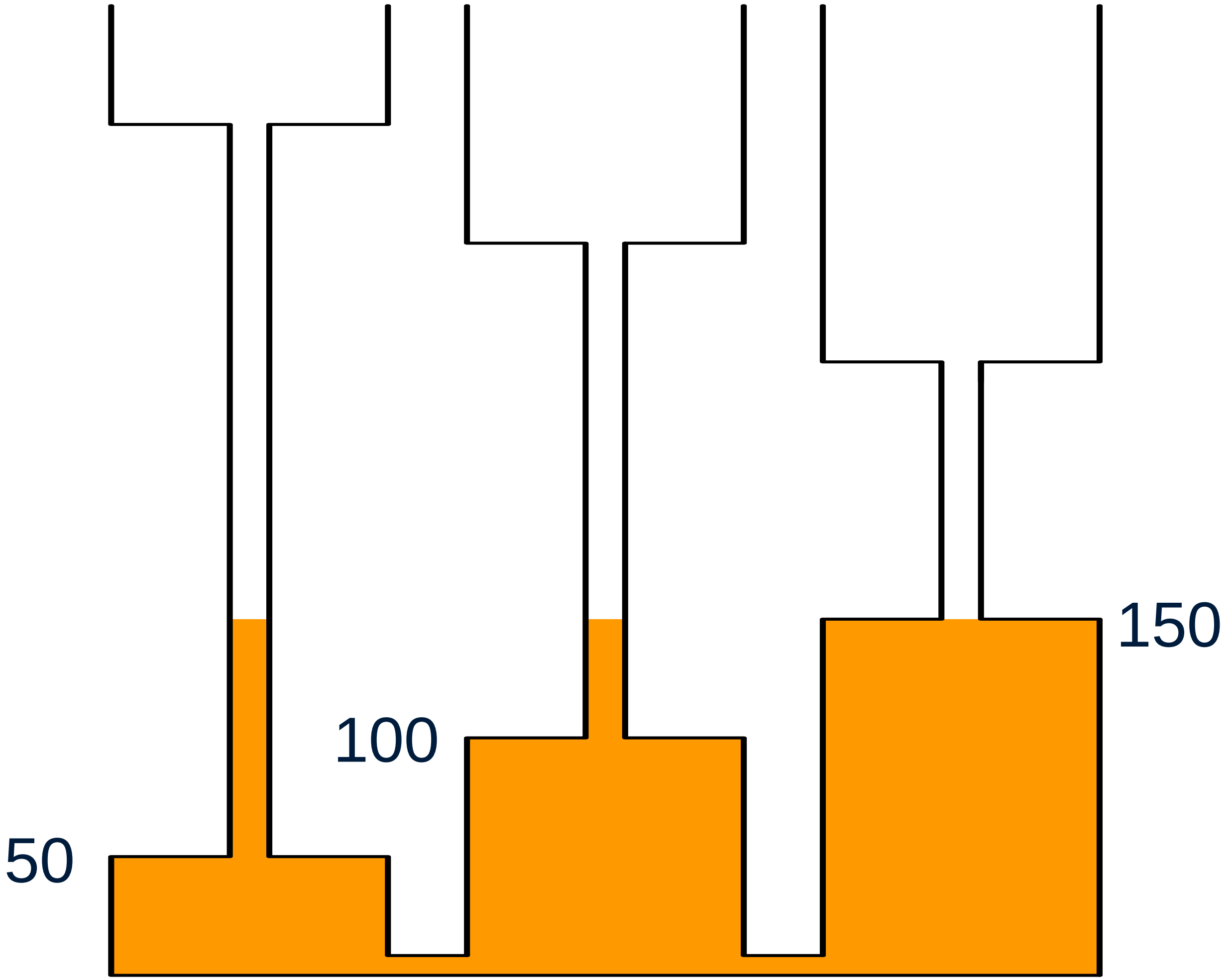
Communicating Vessels: 300



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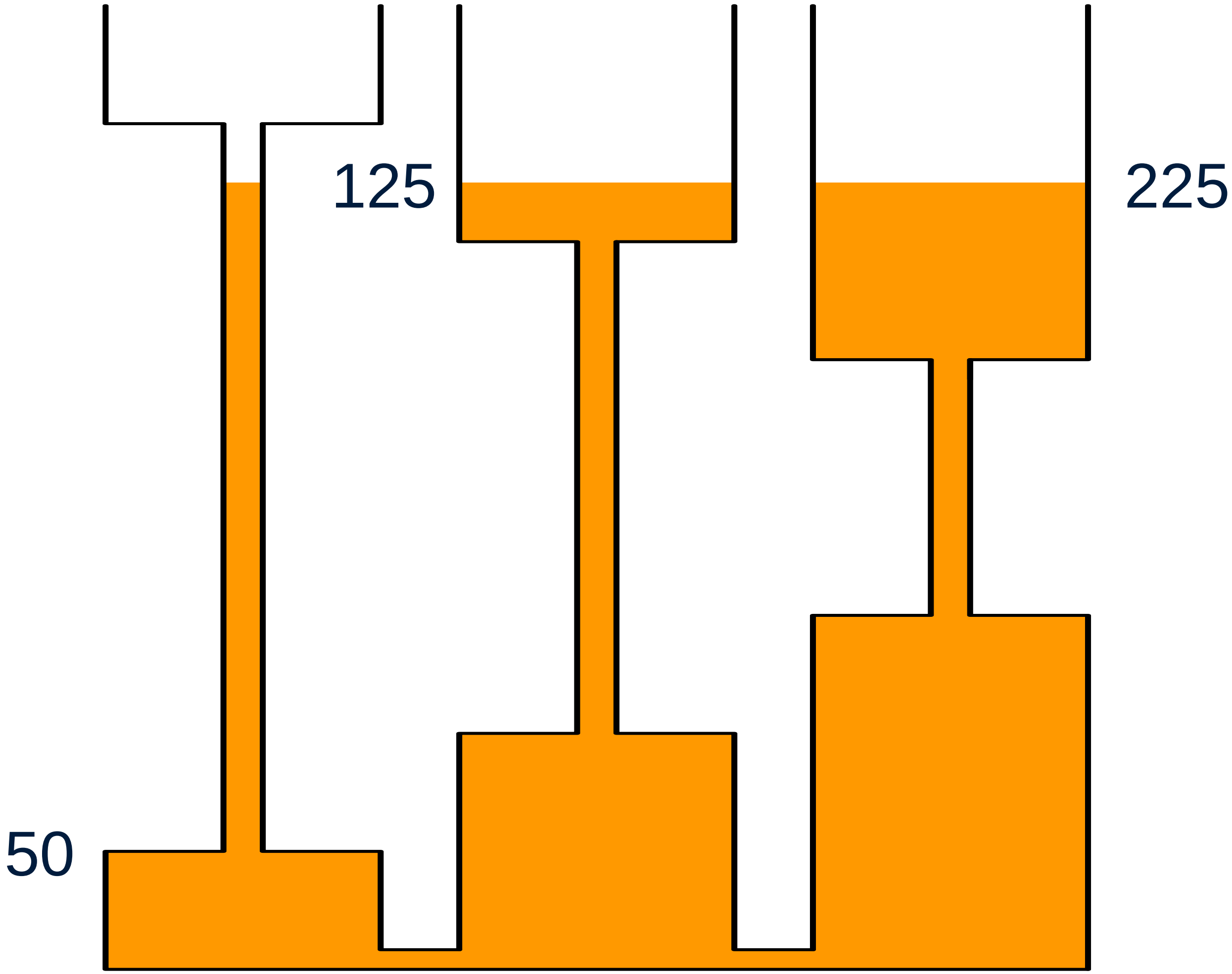
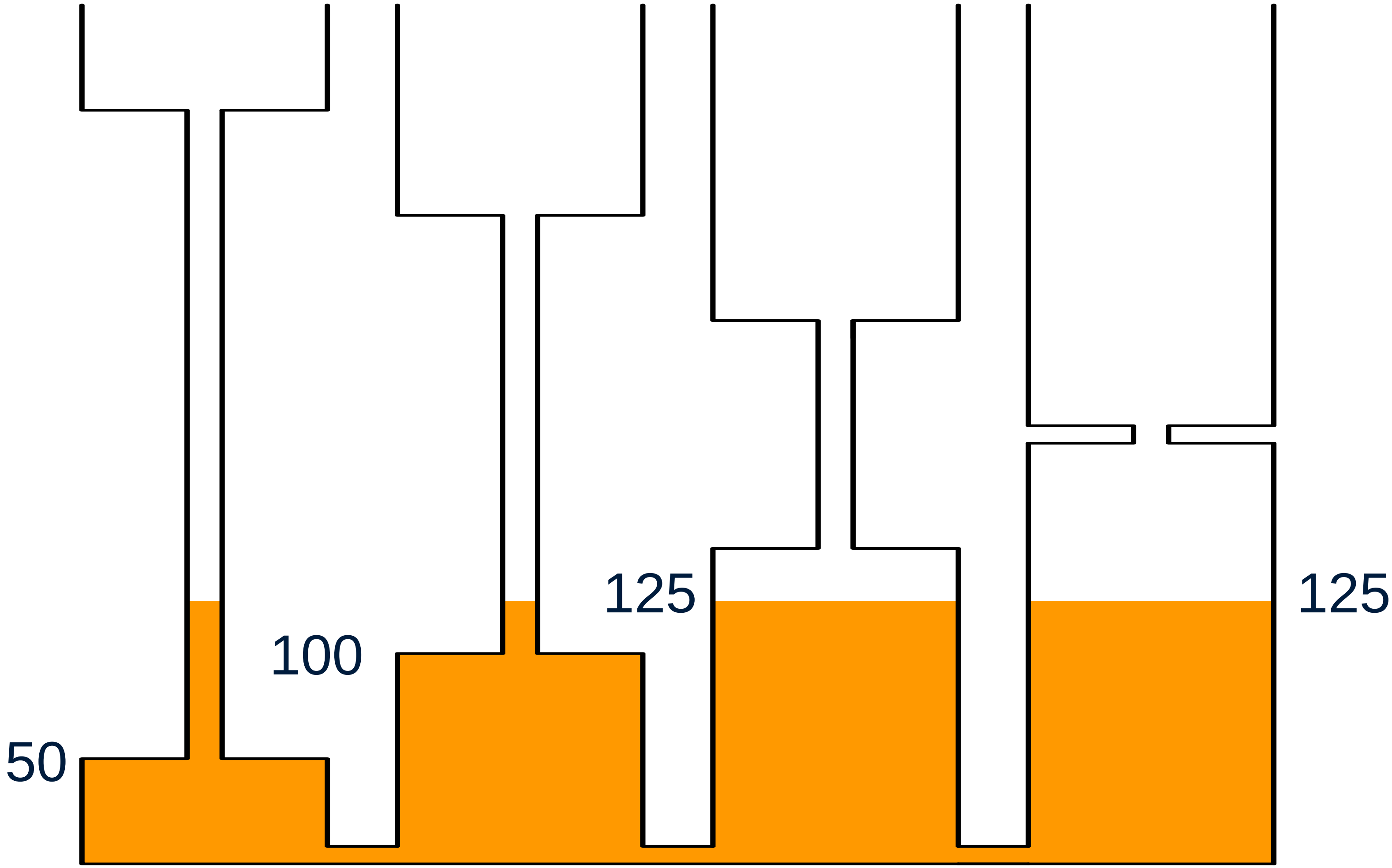
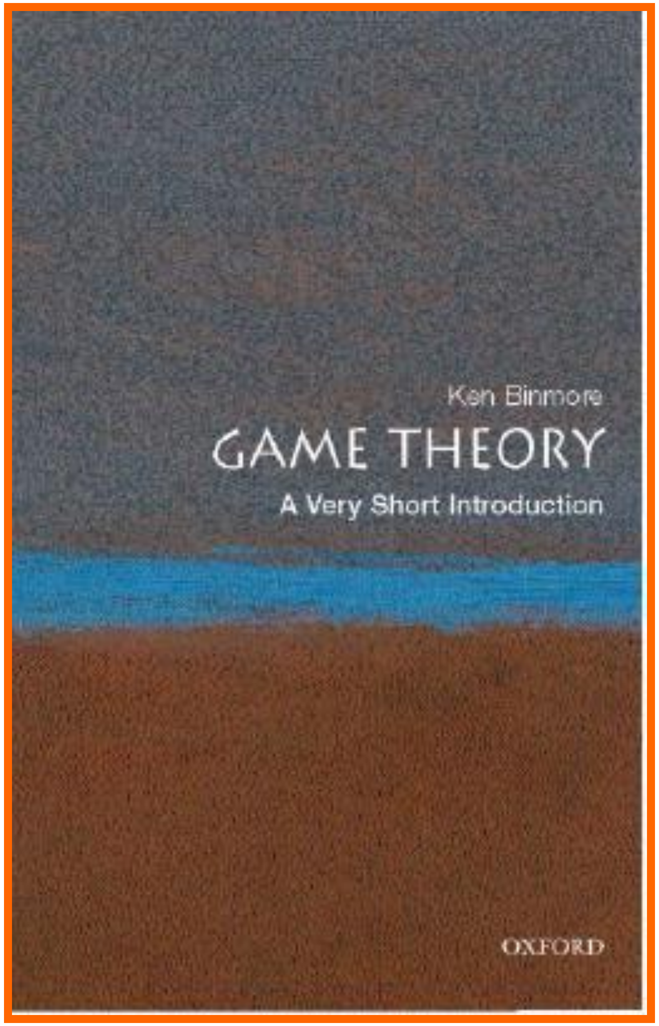
Communicating Vessels: 400



Communicating Vessels: 400 for 4

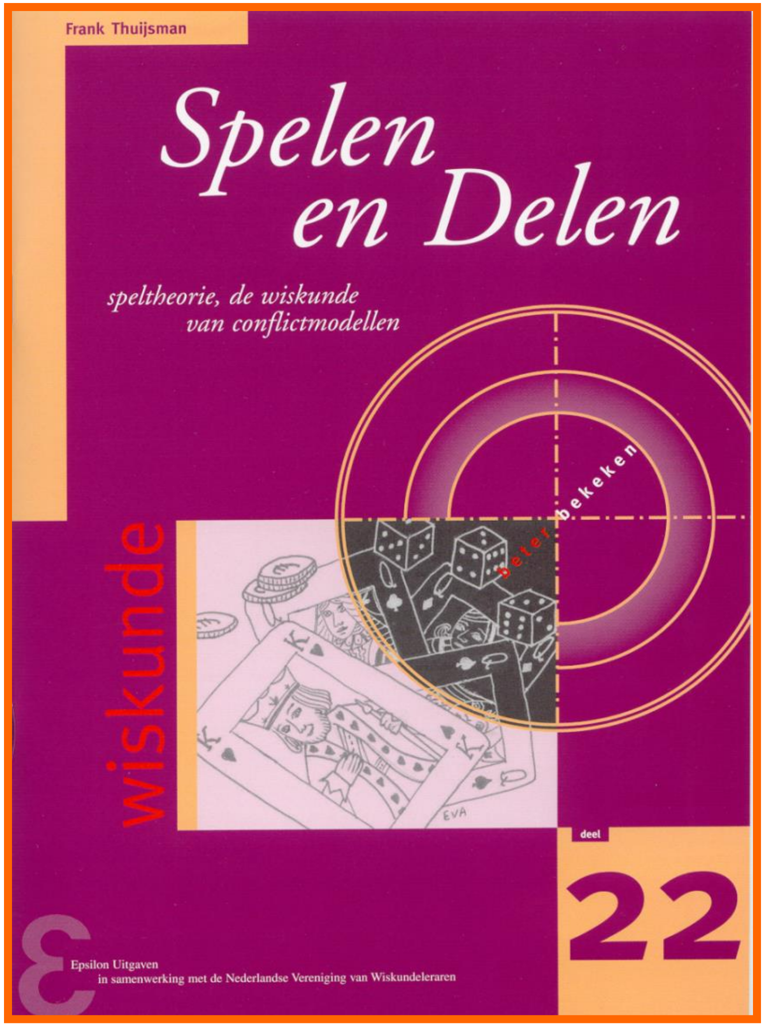


Further Reading



Ken Binmore: ***Game Theory - A very short introduction*,**
Oxford University Press, 2007

- Further Reading



Frank Thuijsman: ***Spelen en Delen*, Epsilon Uitgaven, 2005**

Tobias Harks: Game Theory and Networks (25/03)

Game-Theoretic Model of Selfish Routing

agents minimize $\ell_P(f)$ over $\mathcal{P}_k = \{P : P \text{ is } (s_k, t_k) \text{ path}\}$

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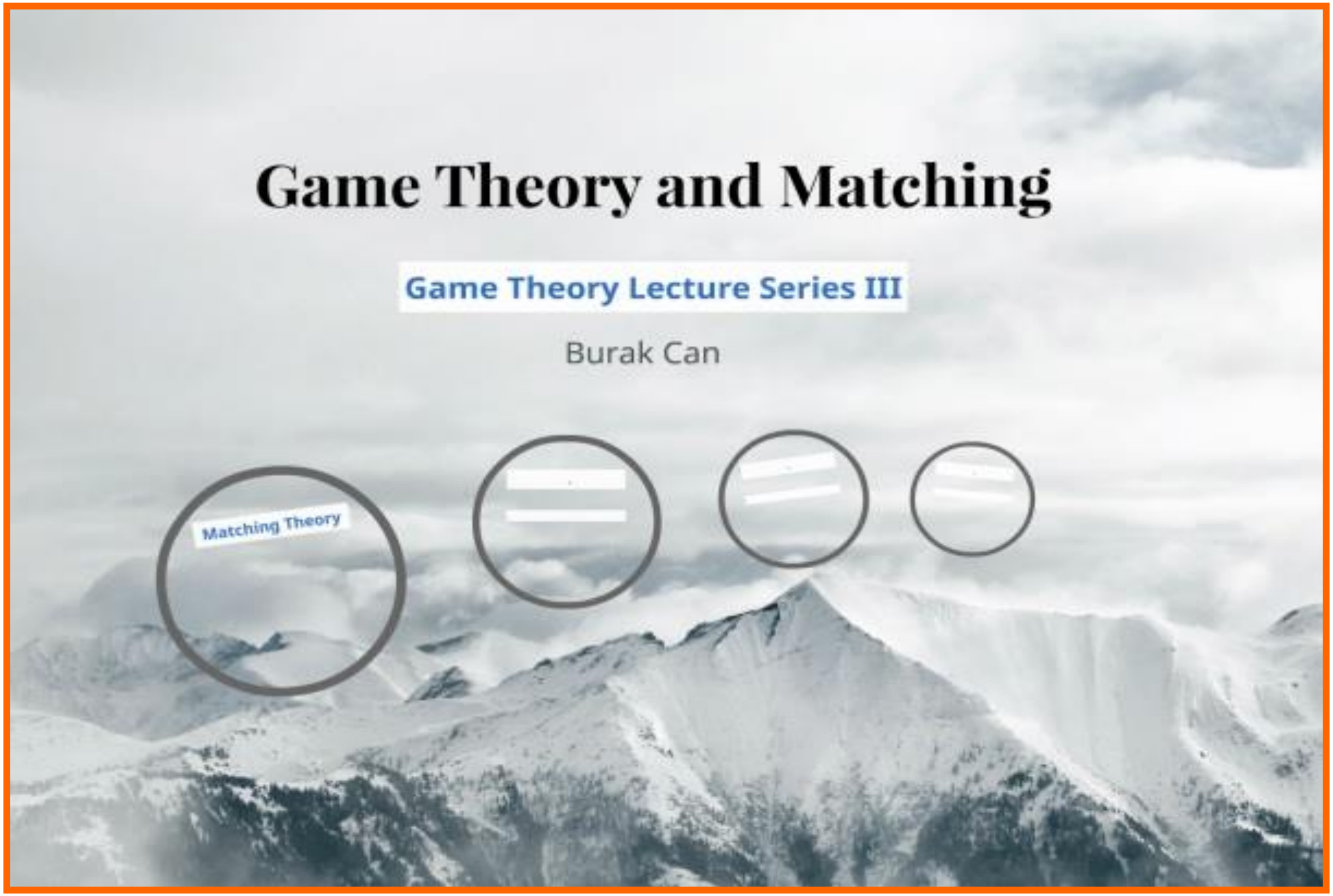
Burak Can: Game Theory and Matching (08/04)



Arno Riedl: Game Theory Experiments (15/04)

Lecture 4 (Arno Riedl, SBE/AE1)
Experimental and Behavioral Game Theory

- **Experimental Game Theory**
  - Uses controlled experiments to study real peoples behavior in situations of conflicts (games)
  - You will participate and make real choices in life experiments
  - Use observed behavior to discuss the reasonableness and empirical validity of important concepts in game theory (dominant strategies, equilibrium)
- **Behavioral Game Theory**
  - Uses insights from experiments, psychology, and neuroscience to build new models of human behavior in situations of conflict
  - Transform game theory from a normative (prescriptive) to a positive (descriptive) theory
  - Models real peoples behavior and allows for bounded rationality, fairness, reciprocity

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Games 2016

5^{th} World Conference of the Game Theory Society

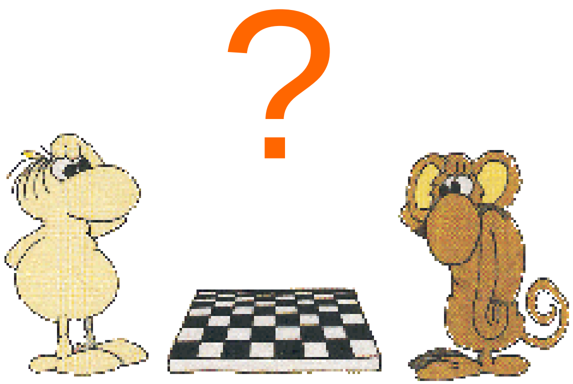
to be held in

Maastricht

July 24 to 28, 2016

f.thuijsman@maastrichtuniversity.nl

Many Thanks for Your Attention!



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