Population Dynamics in Stochastic Games

joint with J. Flesch, P. Uyttendaele, Maastricht University and T. Parthasarathy, Indian Statistical Institute, Chennai

Toulouse, September 12-16, 2011
Outline

1. Introduction
2. Stochastic Games
3. Evolutionary Games
4. Evolutionary Stochastic Games
5. Concluding Remarks

Frank Thuijsman, Maastricht University
Population Dynamics in Stochastic Games
1928, John von Neumann

2-Person Zerosum Games

Existence of Value and Optimal Strategies
1951, John Nash

$n$-Person Non-Zerosum Games

Existence of Equilibria
1953, Lloyd Shapley

2-Person Zerosum Stochastic Games

Existence of Value and Optimal Strategies
Evolutionary Games

- Population of Different Types Playing against Itself.
- Population Distribution $p = (p_1, p_2, \ldots, p_n)$.
- Type $k$ has Fitness $e_k A p$ in Population $p$.
- Concept of Evolutionary Stable Strategies (ESS).

With $b_{ij} = a_{ji}$
How to Model a Population Playing a Stochastic Game?
How to Model a Population Playing a Stochastic Game?

- Some Words about Stochastic Games
How to Model a Population Playing a Stochastic Game?

- Some Words about Stochastic Games
- Some Words about Evolutionary Games
How to Model a Population Playing a Stochastic Game?

- Some Words about Stochastic Games
- Some Words about Evolutionary Games
- Presentation of a Combined Model
The Stochastic Game Model

Finitely Many States, Finitely Many Actions for each Player

Payoffs and Transitions at each Stage 1, 2, 3, 4, ...

Each State can serve as Initial State

Complete Information and Perfect Recall

Discounting or Averaging the Stage Payoffs

Frank Thuijsman, Maastricht University

Population Dynamics in Stochastic Games
The Stochastic Game Model

- Finitely Many States, Finitely Many Actions for each Player
The Stochastic Game Model

- Finitely Many States, Finitely Many Actions for each Player
- Payoffs and Transitions at each Stage 1, 2, 3, 4, ...
The Stochastic Game Model

- Finitely Many States, Finitely Many Actions for each Player
- Payoffs and Transitions at each Stage 1, 2, 3, 4, …
- Each State can serve as Initial State
The Stochastic Game Model

- Finitely Many States, Finitely Many Actions for each Player
- Payoffs and Transitions at each Stage 1, 2, 3, 4, ...
- Each State can serve as Initial State
- Complete Information and Perfect Recall
The Stochastic Game Model

- Finitely Many States, Finitely Many Actions for each Player
- Payoffs and Transitions at each Stage 1, 2, 3, 4, …
- Each State can serve as Initial State
- Complete Information and Perfect Recall
- Discounting or Averaging the Stage Payoffs
Some Highlights of Stochastic Game Theory

\[ a_{ij}(s), b_{ij}(s), p(s, i, j) \]

state 1

\[ y_j(s) \]

state s

state z
Some Highlights of Stochastic Game Theory

1953, L.S. Shapley:  
2-Person Zerosum Stopping Stochastic Games - Value
Some Highlights of Stochastic Game Theory

- 1953, L.S. Shapley: 2-Person Zerosum Stopping Stochastic Games - Value
- 1957, H. Everett / D. Gillette: 2-Person Zerosum Undiscounted Stochastic Games
Some Highlights of Stochastic Game Theory

- 1953, L.S. Shapley: 2-Person Zerosum Stopping Stochastic Games - Value
- 1957, H. Everett / D. Gillette: 2-Person Zerosum Undiscounted Stochastic Games
- 1964, A.M. Fink / M. Takahashi: $n$-Person $\beta$-Discounted Stochastic Games - Equilibria
**Some Highlights of Stochastic Game Theory**

- 1953, L.S. Shapley: 2-Person Zerosum Stopping Stochastic Games - Value
- 1957, H. Everett / D. Gillette: 2-Person Zerosum Undiscounted Stochastic Games
- 1964, A.M. Fink / M. Takahashi: $n$-Person $\beta$-Discounted Stochastic Games - Equilibria
- 1981, J.F. Mertens and A. Neyman: 2-Person Zerosum Undiscounted Stochastic Games - Value
Some Highlights of Stochastic Game Theory

- 1953, L.S. Shapley: 2-Person Zerosum Stopping Stochastic Games - Value
- 1957, H. Everett / D. Gillette: 2-Person Zerosum Undiscounted Stochastic Games
- 1964, A.M. Fink / M. Takahashi: $n$-Person $\beta$-Discounted Stochastic Games - Equilibria
- 1981, J.F. Mertens and A. Neyman: 2-Person Zerosum Undiscounted Stochastic Games - Value
- 2000, N. Vieille: 2-Person Undiscounted Stochastic Games - Equilibria
Evolutionary Games

- Population of Different Types Playing against Itself.
- Population Distribution $p = (p_1, p_2, \ldots, p_n)$.
- Type $k$ has Fitness $e_k Ap$ in Population $p$.
- Concept of Evolutionary Stable Strategies (ESS).

With $b_{ij} = a_{ji}$
The ESS Concept

Evolutionary Games

ESS: Population Distribution $p = (p_1, p_2, \ldots, p_n)$ with

- $pAp \geq qAp \ \forall q$
- If $q \neq p$ and $qAp = pAp$, then $pAq > qAq$

Frank Thuijsman, Maastricht University

Population Dynamics in Stochastic Games
Evolutionary Games

$\mathbf{p}_j$

$p_i$

$a_{ij}, b_{ij}$ with $b_{ij} = a_{ji}$

Population Development by the Replicator Equation:

$\dot{p}_k = p_k (e_k A p - p A p)$
Remarks on ESS and Asymptotic Stability

A Static Concept and a Dynamic Process

ESS Not Always Exists
Replicator Dynamic Not Always Converges
Any ESS is Asymptotically Stable
Limit Points of Dynamic Not Always ESS

Frank Thuijsman, Maastricht University

Population Dynamics in Stochastic Games
Remarks on ESS and Asymptotic Stability

- A Static Concept and a Dynamic Process
Remarks on ESS and Asymptotic Stability

- A Static Concept and a Dynamic Process
- ESS Not Always Exists
Remarks on ESS and Asymptotic Stability

- A Static Concept and a Dynamic Process
- ESS Not Always Exists
- Replicator Dynamic Not Always Converges
Remarks on ESS and Asymptotic Stability

- A Static Concept and a Dynamic Process
- ESS Not Always Exists
- Replicator Dynamic Not Always Converges
- Any ESS is Asymptotically Stable

(Hofbauer and Sigmund, 1998)
Remarks on ESS and Asymptotic Stability

- A Static Concept and a Dynamic Process
- ESS Not Always Exists
- Replicator Dynamic Not Always Converges
- Any ESS is Asymptotically Stable
- Limit Points of Dynamic Not Always ESS
Remarks on ESS and Asymptotic Stability

- A Static Concept and a Dynamic Process
- ESS Not Always Exists
- Replicator Dynamic Not Always Converges
- Any ESS is Asymptotically Stable
- Limit Points of Dynamic Not Always ESS

(Hofbauer and Sigmund, 1998)
Remarks on ESS and Asymptotic Stability

- A Static Concept and a Dynamic Process
- ESS Not Always Exists
- Replicator Dynamic Not Always Converges
- Any ESS is Asymptotically Stable
- Limit Points of Dynamic Not Always ESS

\[
\begin{pmatrix}
0 & 6 & -4 \\
-3 & 0 & 5 \\
-1 & 3 & 0
\end{pmatrix}
\]
(Hofbauer and Sigmund, 1998)
Assumptions for Evolutionary Stochastic Games

\[ a_{ij}(s), b_{ij}(s) \]

\[ p(s, i, j) \]

Symmetric Payoffs:
\[ b_{ij} = a_{ji} \]

Symmetric Transitions:
\[ p(s, i, j) = p(s, j, i) \]

Unichain Stochastic Game

One Ergodic Set for Any Pair of Stationary Strategies
Assumptions for Evolutionary Stochastic Games

- Symmetric Payoffs: \( b_{ij} = a_{ji} \)
Assumptions for Evolutionary Stochastic Games

- Symmetric Payoffs: $b_{ij} = a_{ji}$
- Symmetric Transitions: $p(s, i, j) = p(s, j, i)$
Assumptions for Evolutionary Stochastic Games

- Symmetric Payoffs: $b_{ij} = a_{ji}$
- Symmetric Transitions: $p(s, i, j) = p(s, j, i)$
- Unichain Stochastic Game
Assumptions for Evolutionary Stochastic Games

- Symmetric Payoffs: \( b_{ij} = a_{ji} \)
- Symmetric Transitions: \( p(s, i, j) = p(s, j, i) \)
- Unichain Stochastic Game
  One Ergodic Set for Any Pair of Stationary Strategies
Assumptions for Evolutionary Stochastic Games

- Symmetric Payoffs: $b_{ij} = a_{ji}$
- Symmetric Transitions: $p(s, i, j) = p(s, j, i)$
- Unichain Stochastic Game
  - One Ergodic Set for Any Pair of Stationary Strategies
- Types Correspond to Pure Stationary Strategies
Assumptions Continued

\[ \begin{align*}
\cdots x_i(s) \quad & a_{ij}(s), b_{ij}(s) & \cdots p(s, i, j) \\
\text{state 1} \quad & \text{state } s \quad & \text{state } z
\end{align*} \]
Assumptions Continued

- Fitness of Type $k$ in Population $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$ is Average Reward $\gamma(e_k, x)$ where $x$ Stationary Strategy determined by $\bar{x}$
Assumptions Continued

Fitness of Type $k$ in Population $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$ is Average Reward $\gamma(e_k, x)$ where $x$ Stationary Strategy determined by $\bar{x}$

Different Populations can give Same Stationary Strategy
**Assumptions Continued**

- Fitness of Type $k$ in Population $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$ is Average Reward $\gamma(e_k, x)$ where $x$ Stationary Strategy determined by $\bar{x}$
- Different Populations can give Same Stationary Strategy
- Stationary Strategy $x$ is ESS if
  - $\gamma(x, x) \geq \gamma(y, x)$ $\forall$ Stationary Strategies $y$
  - If $y \neq x$ and $\gamma(y, x) = \gamma(x, x)$, then $\gamma(x, y) > \gamma(y, y)$

---

Frank Thuijsman, Maastricht University

Population Dynamics in Stochastic Games
Assumptions Continued

- Fitness of Type $k$ in Population $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$ is Average Reward $\gamma(e_k, x)$ where $x$ Stationary Strategy determined by $\bar{x}$
- Different Populations can give Same Stationary Strategy
- Stationary Strategy $x$ is ESS if
  - $\gamma(x, x) \geq \gamma(y, x)$ $\forall$ Stationary Strategies $y$
  - If $y \neq x$ and $\gamma(y, x) = \gamma(x, x)$, then $\gamma(x, y) > \gamma(y, y)$
- Population Development by Replicator Dynamic
  - $\dot{x}_k = \bar{x}_k (\gamma(e_k, x) - \gamma(x, x))$
Some Remarks

- ESS Not Always Exists
- Replicator Dynamic Not Always Converges
- Limit Points of Dynamic Not Always give ESS

Frank Thuijsman, Maastricht University
Population Dynamics in Stochastic Games
Some Remarks

ESS Not Always Exists
Some Remarks

- ESS Not Always Exists
- Replicator Dynamic Not Always Converges
Some Remarks

- ESS Not Always Exists
- Replicator Dynamic Not Always Converges
- Limit Points of Dynamic Not Always give ESS
Why Unichain?

Frank Thuijsman, Maastricht University
Population Dynamics in Stochastic Games
A 2 State Example with Replicator Dynamics

<table>
<thead>
<tr>
<th>State 1</th>
<th>State 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 1 \ (0, 1)</td>
<td>4, 3 \ (.5, .5)</td>
</tr>
<tr>
<td>3, 4 \ (.5, .5)</td>
<td>2, 2 \ (0, 1)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>State 1</th>
<th>State 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>3, 3 \ (1, 0)</td>
<td>5, 4 \ (.5, .5)</td>
</tr>
<tr>
<td>4, 5 \ (.5, .5)</td>
<td>2, 2 \ (1, 0)</td>
</tr>
</tbody>
</table>

Frank Thuijsman, Maastricht University
Population Dynamics in Stochastic Games
A 2 State Example with Replicator Dynamics

<table>
<thead>
<tr>
<th>State 1</th>
<th>State 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 1</td>
<td>3, 3</td>
</tr>
<tr>
<td>(0, 1)</td>
<td>(.5, .5)</td>
</tr>
<tr>
<td>3, 4</td>
<td>4, 5</td>
</tr>
<tr>
<td>(.5, .5)</td>
<td>(.5, .5)</td>
</tr>
<tr>
<td>4, 3</td>
<td>2, 2</td>
</tr>
<tr>
<td>(.5, .5)</td>
<td>(0, 1)</td>
</tr>
</tbody>
</table>

Trajectory

Frank Thuijsman, Maastricht University
Population Dynamics in Stochastic Games
A 2 State Example with Replicator Dynamics

<table>
<thead>
<tr>
<th>State 1</th>
<th>State 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 1</td>
<td>3, 4</td>
</tr>
<tr>
<td>(0, 1)</td>
<td>(.5, .5)</td>
</tr>
<tr>
<td>3, 4</td>
<td>2, 2</td>
</tr>
<tr>
<td>(.5, .5)</td>
<td>(0, 1)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>State 1</th>
<th>State 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>4, 3</td>
<td>5, 4</td>
</tr>
<tr>
<td>(.5, .5)</td>
<td>(.5, .5)</td>
</tr>
<tr>
<td>2, 2</td>
<td>2, 2</td>
</tr>
<tr>
<td>(0, 1)</td>
<td>(1, 0)</td>
</tr>
</tbody>
</table>
A 2 State Example with Replicator Dynamics

<table>
<thead>
<tr>
<th>State 1</th>
<th>1, 1</th>
<th>4, 3</th>
<th>(0, 1)</th>
<th>(.5, .5)</th>
<th>3, 4</th>
<th>2, 2</th>
<th>(.5, .5)</th>
<th>1, 1</th>
<th>4, 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>State 2</td>
<td>3, 3</td>
<td>5, 4</td>
<td>(1, 0)</td>
<td>(.5, .5)</td>
<td>4, 5</td>
<td>2, 2</td>
<td>(.5, .5)</td>
<td>3, 3</td>
<td>5, 4</td>
</tr>
</tbody>
</table>

Frank Thuijsman, Maastricht University
Population Dynamics in Stochastic Games
The Fictitious Play Process:
Playing Best Replies against Observed Action Frequencies
The Fictitious Play Process:
Playing Best Replies against Observed Action Frequencies
- For Matrix Games FP leads to Optimal Strategies
1951, George Brown / Julia Robinson

The Fictitious Play Process:
Playing Best Replies against Observed Action Frequencies
- For Matrix Games FP leads to Optimal Strategies
- No FP Convergence for Bimatrix Games (Shapley, 1964)
The Fictitious Play Process:
Playing Best Replies against Observed Action Frequencies
- For Matrix Games FP leads to Optimal Strategies
- No FP Convergence for Bimatrix Games (Shapley, 1964)
- ...
A 2 State Example with Fictitious Play

<table>
<thead>
<tr>
<th>State 1</th>
<th></th>
<th>State 2</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 1</td>
<td>(0, 1)</td>
<td>3, 3</td>
<td>(1, 0)</td>
</tr>
<tr>
<td>3, 4</td>
<td>(.5, .5)</td>
<td>4, 5</td>
<td>(.5, .5)</td>
</tr>
<tr>
<td>2, 2</td>
<td>(0, 1)</td>
<td>2, 2</td>
<td>(1, 0)</td>
</tr>
</tbody>
</table>

Frank Thuijsman, Maastricht University
Population Dynamics in Stochastic Games
A 2 State Example with Fictitious Play

<table>
<thead>
<tr>
<th>State 1</th>
<th>1, 1</th>
<th>4, 3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(0, 1)</td>
<td>(.5, .5)</td>
</tr>
<tr>
<td>State 2</td>
<td>3, 4</td>
<td>2, 2</td>
</tr>
<tr>
<td></td>
<td>(.5, .5)</td>
<td>(0, 1)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>State 1</th>
<th>3, 3</th>
<th>5, 4</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(1, 0)</td>
<td>(.5, .5)</td>
</tr>
<tr>
<td>State 2</td>
<td>4, 5</td>
<td>2, 2</td>
</tr>
<tr>
<td></td>
<td>(.5, .5)</td>
<td>(1, 0)</td>
</tr>
</tbody>
</table>

Frank Thuijsman, Maastricht University
Population Dynamics in Stochastic Games
## A 3 State Example with Replicator Dynamics

<table>
<thead>
<tr>
<th>State</th>
<th>Payoff 1</th>
<th>Payoff 2</th>
<th>Payoff 3</th>
<th>Payoff 4</th>
<th>Payoff 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 1</td>
<td>1, 1</td>
<td>4, 3</td>
<td>3, 4</td>
<td>1, 1</td>
<td>4, 3</td>
</tr>
<tr>
<td></td>
<td>(.5, 0, .5)</td>
<td>(.5, .5, 0)</td>
<td>(.5, .5, 0)</td>
<td>(.5, 0, .5)</td>
<td>(.5, .5, 0)</td>
</tr>
<tr>
<td>3, 3</td>
<td>3, 3</td>
<td>5, 4</td>
<td>4, 5</td>
<td>3, 3</td>
<td>5, 4</td>
</tr>
<tr>
<td></td>
<td>(1, 0, 0)</td>
<td>(.5, 0, .5)</td>
<td>(.5, 0, .5)</td>
<td>(1, 0, 0)</td>
<td>(.5, 0, .5)</td>
</tr>
<tr>
<td>4, 4</td>
<td>4, 4</td>
<td>6, 7</td>
<td>7, 6</td>
<td>4, 4</td>
<td>6, 7</td>
</tr>
<tr>
<td></td>
<td>(0, 1, 0)</td>
<td>(0, .5, .5)</td>
<td>(0, .5, .5)</td>
<td>(0, 1, 0)</td>
<td>(0, .5, .5)</td>
</tr>
</tbody>
</table>

- **State 1**:
  - Transition probabilities:
    - From state 1: (0, 1, 0) to state 1
    - From state 2: (.5, .5, 0) to state 3

- **State 2**:
  - Transition probabilities:
    - From state 1: (.5, .5, 0) to state 2
    - From state 2: (0, 1) to state 2

- **State 3**:
  - Transition probabilities:
    - From state 1: (.5, .5, 0) to state 3
    - From state 2: (0, 0, 1) to state 1

Frank Thuijsman, Maastricht University

Population Dynamics in Stochastic Games
## A 3 State Example with Replicator Dynamics

<table>
<thead>
<tr>
<th>State 1</th>
<th>State 2</th>
<th>State 3</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>1, 1</strong></td>
<td>4, 3</td>
<td><strong>4, 4</strong></td>
</tr>
<tr>
<td>(.5, 0, .5)</td>
<td>(.5, .5, 0)</td>
<td>(0, 1, 0)</td>
</tr>
<tr>
<td><strong>3, 4</strong></td>
<td>2, 2</td>
<td><strong>6, 7</strong></td>
</tr>
<tr>
<td>(.5, .5, 0)</td>
<td>(0, .5, .5)</td>
<td>(0, .5, .5)</td>
</tr>
<tr>
<td><strong>3, 3</strong></td>
<td>5, 4</td>
<td><strong>7, 6</strong></td>
</tr>
<tr>
<td>(1, 0, 0)</td>
<td>(.5, 0, .5)</td>
<td>(0, 1, 0)</td>
</tr>
</tbody>
</table>

Frank Thuijsman, Maastricht University

Population Dynamics in Stochastic Games
A 3 State Example with Fictitious Play

<table>
<thead>
<tr>
<th>State 1</th>
<th>State 2</th>
<th>State 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 1</td>
<td>3, 3</td>
<td>4, 4</td>
</tr>
<tr>
<td>(.5, 0, .5)</td>
<td>(1, 0, 0)</td>
<td>(0, 1, 0)</td>
</tr>
<tr>
<td>4, 3</td>
<td>5, 4</td>
<td>6, 7</td>
</tr>
<tr>
<td>(.5, .5, 0)</td>
<td>(.5, 0, .5)</td>
<td>(0, .5, .5)</td>
</tr>
<tr>
<td>3, 4</td>
<td>4, 5</td>
<td>7, 6</td>
</tr>
<tr>
<td>(.5, .5, 0)</td>
<td>(.5, 0, .5)</td>
<td>(0, .5, .5)</td>
</tr>
<tr>
<td>2, 2</td>
<td>2, 2</td>
<td>5, 5</td>
</tr>
<tr>
<td>(0, .5, .5)</td>
<td>(0, 0, 1)</td>
<td>(1, 0, 0)</td>
</tr>
</tbody>
</table>

Frank Thuijsman, Maastricht University
Population Dynamics in Stochastic Games
A 3 State Example with Fictitious Play

<table>
<thead>
<tr>
<th>State 1</th>
<th>State 2</th>
<th>State 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 1</td>
<td>3, 3</td>
<td>4, 4</td>
</tr>
<tr>
<td>(.5, 0, .5)</td>
<td>(.1, 0, 0)</td>
<td>(.0, .5, .5)</td>
</tr>
<tr>
<td>3, 4</td>
<td>4, 5</td>
<td>6, 7</td>
</tr>
<tr>
<td>(.5, .5, 0)</td>
<td>(.5, 0, .5)</td>
<td>(.0, .5, .5)</td>
</tr>
<tr>
<td>2, 2</td>
<td>2, 2</td>
<td>5, 5</td>
</tr>
<tr>
<td>(0, .5, .5)</td>
<td>(0, 0, 1)</td>
<td>(1, 0, 0)</td>
</tr>
</tbody>
</table>

Trajectory

Frank Thuijsman, Maastricht University
Population Dynamics in Stochastic Games
About this Model

- Existence of Symmetric Equilibria for Symmetric Stochastic Games?
About this Model

- Existence of Symmetric Equilibria for Symmetric Stochastic Games?
- Relation between Replicator Dynamic and Fictitious Play for Symmetric Stochastic Games?
About this Model

- Existence of Symmetric Equilibria for Symmetric Stochastic Games?
- Relation between Replicator Dynamic and Fictitious Play for Symmetric Stochastic Games?
- Some Stability Issues on Population Dynamic
Other ‘Evolutionary’ Work in Maastricht
Other ‘Evolutionary’ Work in Maastricht

Examining the effects of periodic fitness in replicator dynamics
Other ‘Evolutionary’ Work in Maastricht

Examining the effects of local replicator dynamics

Frank Thuijsman, Maastricht University
Population Dynamics in Stochastic Games
Other ‘Evolutionary’ Work in Maastricht

Studying sex choice ovipositioning behavior of parasitoid wasps
Thank you for your attention!
Any comment is welcome!

This presentation will be available at
www.personeel.unimaas.nl/F-Thuijsman