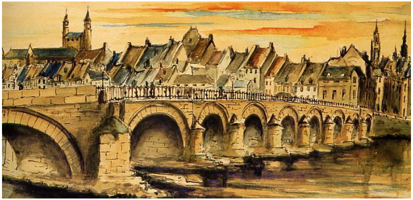
Population Dynamics in Stochastic Games



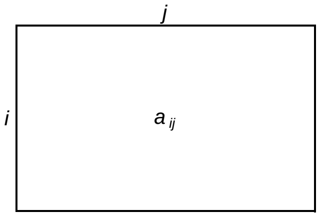
joint with J. Flesch, P. Uyttendaele, Maastricht University
and T. Parthasarathy, Indian Statistical Institute, Chennai

Toulouse, September 12-16, 2011

Outline

1928, John von Neumann

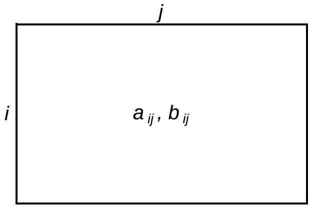
2-Person Zerosum Games



Existence of Value and Optimal Strategies

1951, John Nash

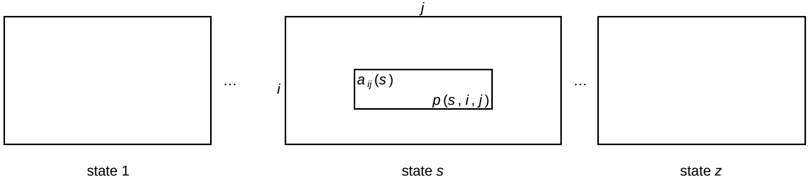
n-Person Non-Zerosum Games



Existence of Equilibria

1953, Lloyd Shapley

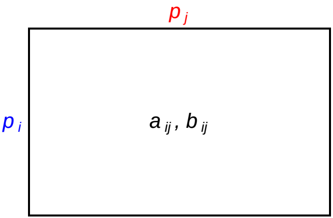
2-Person Zerosum Stochastic Games



Existence of Value and Optimal Strategies

1973, John Maynard Smith and George Price

Evolutionary Games



with $b_{ij} = a_{ji}$

- Population of Different Types Playing against Itself.
- Population Distribution $p = (p_1, p_2, \ldots, p_n)$.
- Type k has Fitness $e_k Ap$ in Population p.
- Concept of Evolutionary Stable Strategies (ESS).

Question

How to Model a Population Playing a Stochastic Game?

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- Some Words about Stochastic Games

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- Some Words about Stochastic Games
- Some Words about Evolutionary Games

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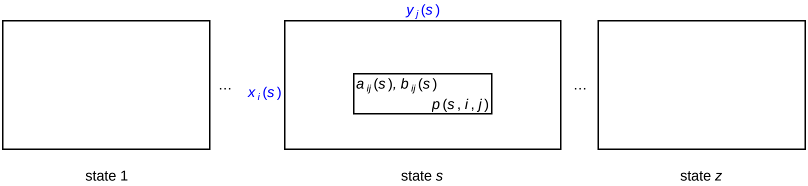
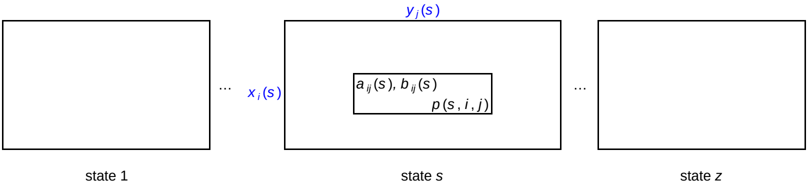
How to Model a Population Playing a Stochastic Game?

- Some Words about Stochastic Games
- Some Words about Evolutionary Games
- Presentation of a Combined Model

The Stochastic Game Model

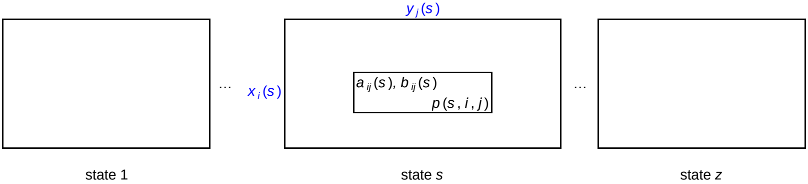


The Stochastic Game Model



- Finitely Many States, Finitely Many Actions for each Player

The Stochastic Game Model



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- Payoffs and Transitions at each Stage $1, 2, 3, 4, \ldots$

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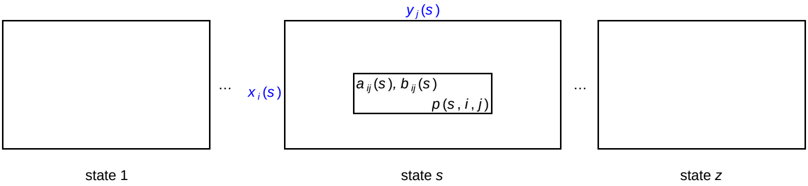

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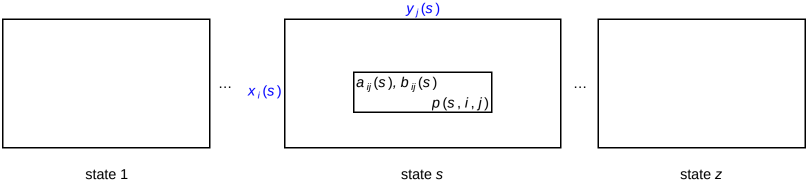

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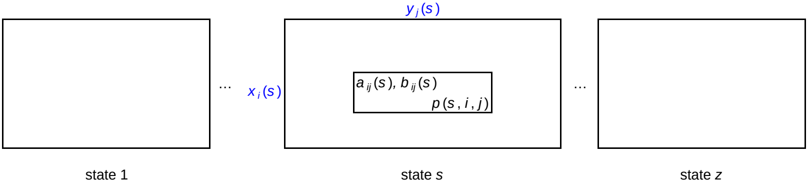
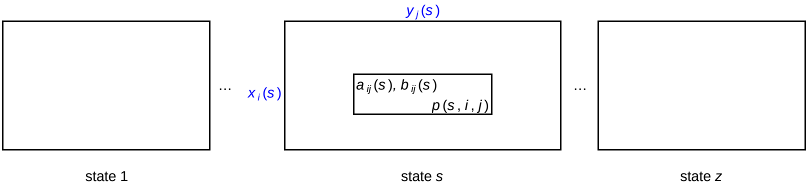


- Finitely Many States, Finitely Many Actions for each Player
- Payoffs and Transitions at each Stage $1, 2, 3, 4, \ldots$
- Each State can serve as Initial State
- Complete Information and Perfect Recall
- Discounting or Averaging the Stage Payoffs

Some Highlights of Stochastic Game Theory



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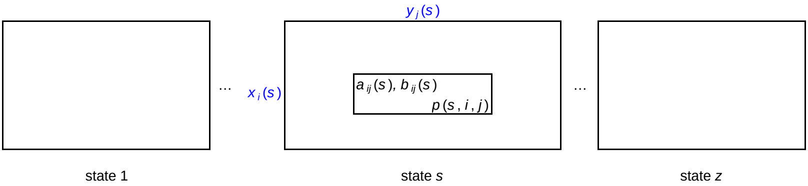

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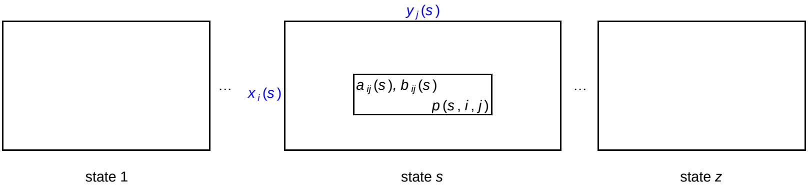

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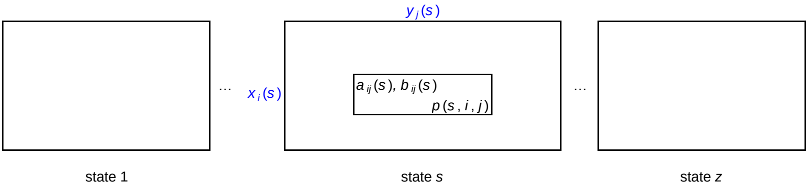

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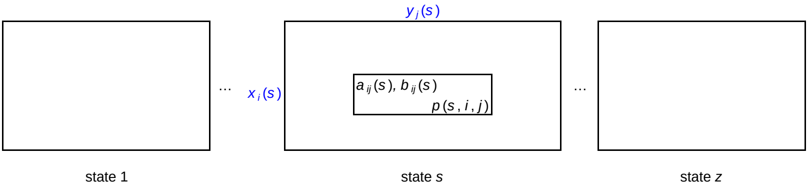

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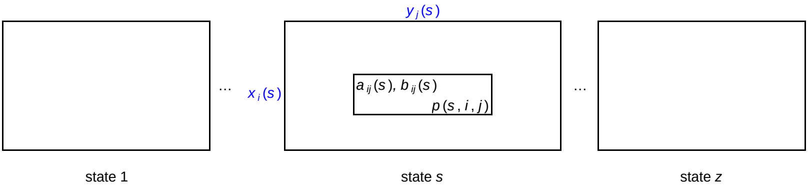

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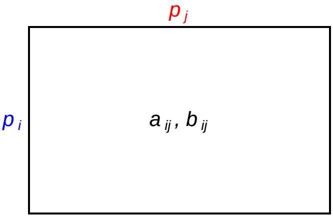
Evolutionary Games



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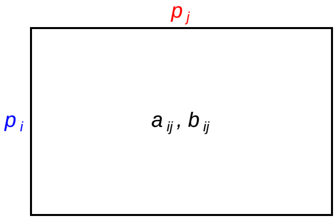
The ESS Concept

Evolutionary Games



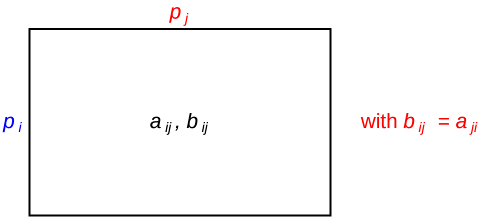
with $b_{ij} = a_{ji}$

ESS: Population Distribution $p = (p_1, p_2, \ldots, p_n)$ with

- $pAp \geq qAp \;\; \forall q$
- If $q \neq p$ and $qAp = pAp$, then $pAq > qAq$

The Replicator Dynamic by Taylor and Jonker, 1978

Evolutionary Games



Population Development by the Replicator Equation:

- $\dot{p}_k = p_k \left(e_k Ap - pAp \right)$

Remarks on ESS and Asymptotic Stability

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- A Static Concept and a Dynamic Process

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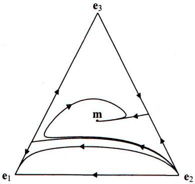

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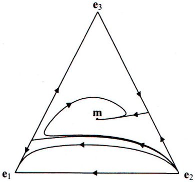
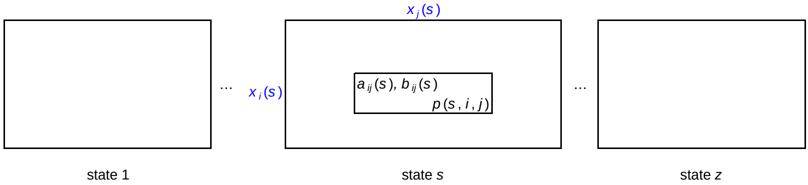


$$\begin{pmatrix} 0 & 6 & -4 \\ -3 & 0 & 5 \\ -1 & 3 & 0 \end{pmatrix}$$ (Hofbauer and Sigmund, 1998)

Assumptions for Evolutionary Stochastic Games



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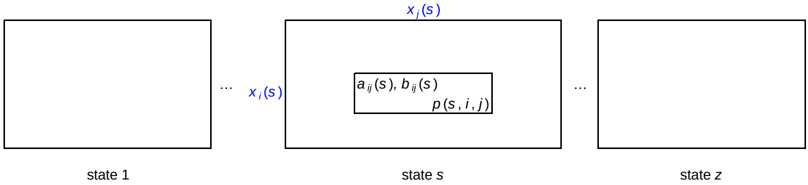

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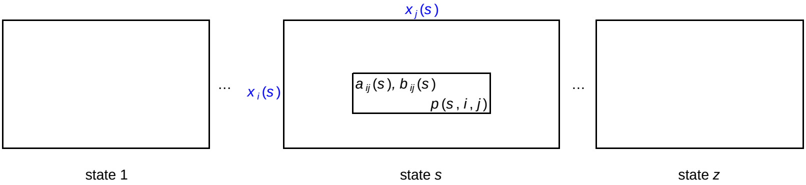

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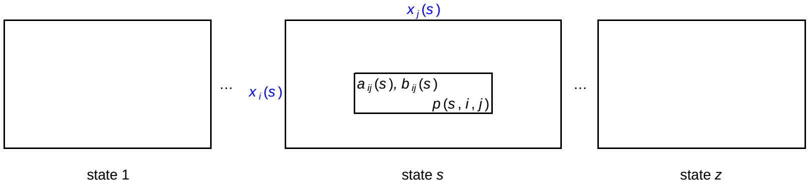

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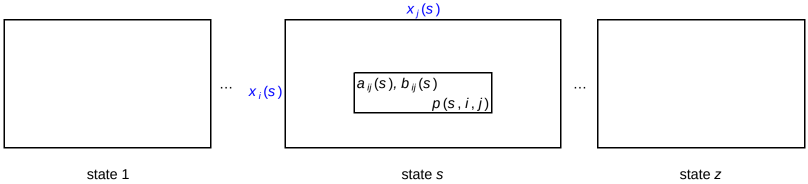

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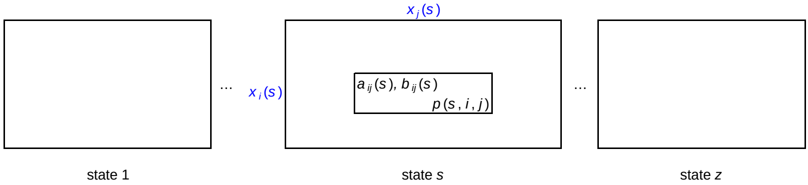

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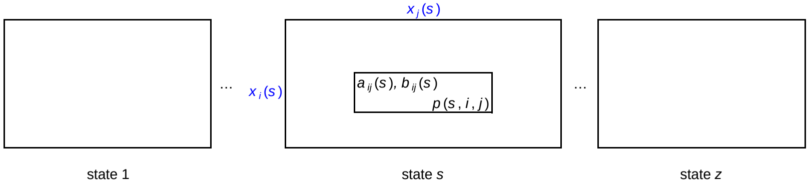
- Types Correspond to Pure Stationary Strategies

Assumptions Continued



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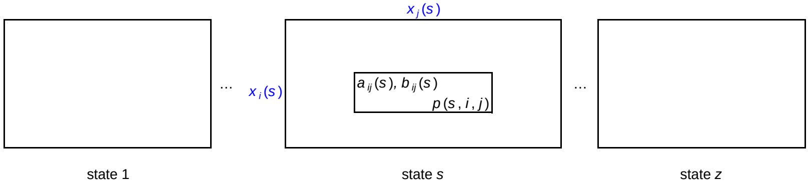
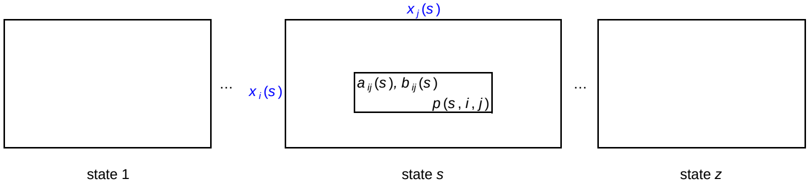


- Fitness of Type k in Population $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$ is Average Reward $\gamma(e_k, x)$ where x Stationary Strategy determined by $\bar{x}$

Assumptions Continued



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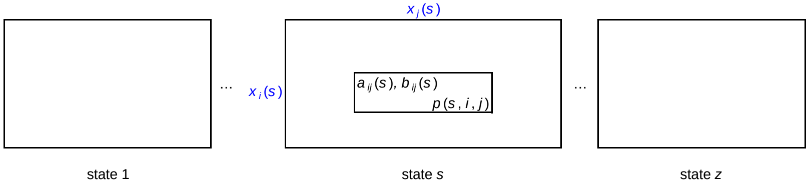

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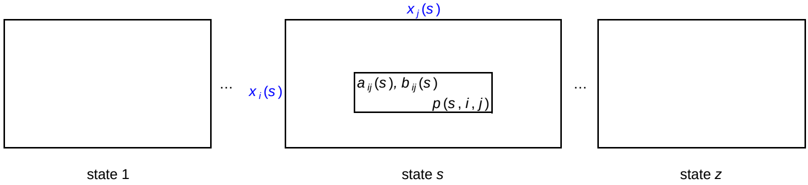
Assumptions Continued



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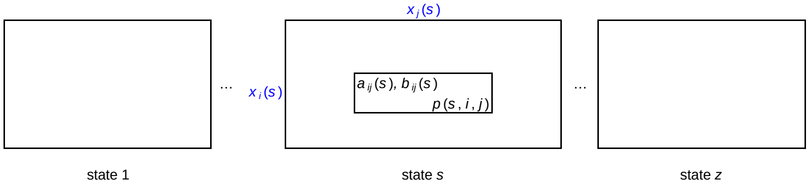
  - If $y \neq x$ and $\gamma(y, x) = \gamma(x, x)$, then $\gamma(x, y) > \gamma(y, y)$
- Population Development by Replicator Dynamic
  - $\dot{\bar{x}}_k = \bar{x}_k \, (\gamma(e_k, x) - \gamma(x, x))$

Some Remarks



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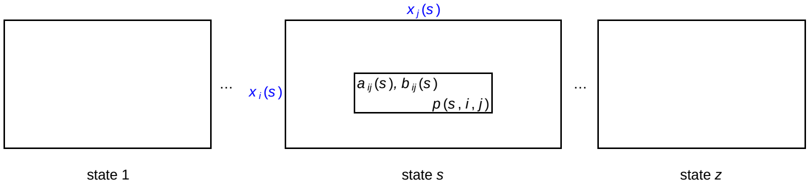

- ESS Not Always Exists

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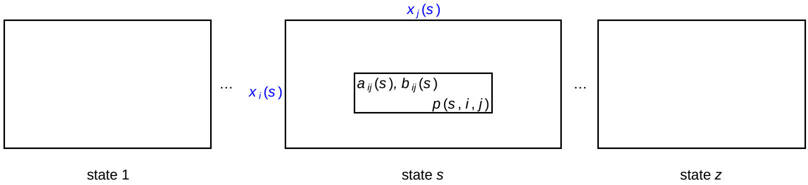

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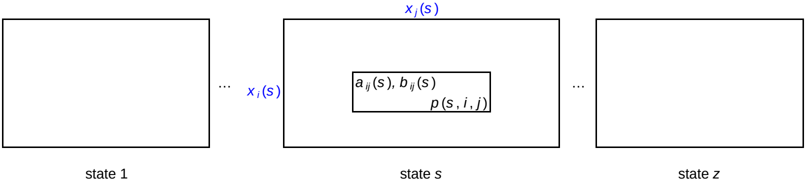

- ESS Not Always Exists
- Replicator Dynamic Not Always Converges
- Limit Points of Dynamic Not Always give ESS

Why Unichain?

3, 3 (1, 0)	5, 4 (1, 0)
4, 5 (1, 0)	2, 2 (0, 1)

state 1

2, 2 (0, 1)

state 2

A 2 State Example with Replicator Dynamics

1, 1 (0, 1)	4, 3 (.5, .5)
3, 4 (.5, .5)	2, 2 (0, 1)

state 1

3, 3 (1, 0)	5, 4 (.5, .5)
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(Trajectory)

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state 1

State 2:

3, 3 (1, 0)	5, 4 (.5, .5)
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state 2

(Trajectory)

1951, George Brown / Julia Robinson

The Fictitious Play Process:
Playing Best Replies against Observed Action Frequencies

1951, George Brown / Julia Robinson

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- For Matrix Games FP leads to Optimal Strategies

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A 2 State Example with Fictitious Play

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state 1

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state 2

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state 1

3, 3 (1, 0)	5, 4 (.5, .5)
4, 5 (.5, .5)	2, 2 (1, 0)

state 2

(Trajectory)

A 3 State Example with Replicator Dynamics

1, 1 (.5, 0, .5)	4, 3 (.5, .5, 0)
3, 4 (.5, .5, 0)	2, 2 (0, .5, .5)

state 1

3, 3 (1, 0, 0)	5, 4 (.5, 0, .5)
4, 5 (.5, 0, .5)	2, 2 (0, 0, 1)

state 2

4, 4 (0, 1, 0)	6, 7 (0, .5, .5)
7, 6 (0, .5, .5)	5, 5 (1, 0, 0)

state 3

A 3 State Example with Replicator Dynamics

1, 1 (.5, 0, .5)	4, 3 (.5, .5, 0)
3, 4 (.5, .5, 0)	2, 2 (0, .5, .5)

state 1

3, 3 (1, 0, 0)	5, 4 (.5, 0, .5)
4, 5 (.5, 0, .5)	2, 2 (0, 0, 1)

state 2

4, 4 (0, 1, 0)	6, 7 (0, .5, .5)
7, 6 (0, .5, .5)	5, 5 (1, 0, 0)

state 3

(Trajectory)

A 3 State Example with Fictitious Play

1, 1 (.5, 0, .5)	4, 3 (.5, .5, 0)
3, 4 (.5, .5, 0)	2, 2 (0, .5, .5)

state 1

3, 3 (1, 0, 0)	5, 4 (.5, 0, .5)
4, 5 (.5, 0, .5)	2, 2 (0, 0, 1)

state 2

4, 4 (0, 1, 0)	6, 7 (0, .5, .5)
7, 6 (0, .5, .5)	5, 5 (1, 0, 0)

state 3

A 3 State Example with Fictitious Play

1, 1 (.5, 0, .5)	4, 3 (.5, .5, 0)
3, 4 (.5, .5, 0)	2, 2 (0, .5, .5)

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4, 5 (.5, 0, .5)	2, 2 (0, 0, 1)

state 2

4, 4 (0, 1, 0)	6, 7 (0, .5, .5)
7, 6 (0, .5, .5)	5, 5 (1, 0, 0)

state 3

(Trajectory)

About this Model

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Other ‘Evolutionary’ Work in Maastricht

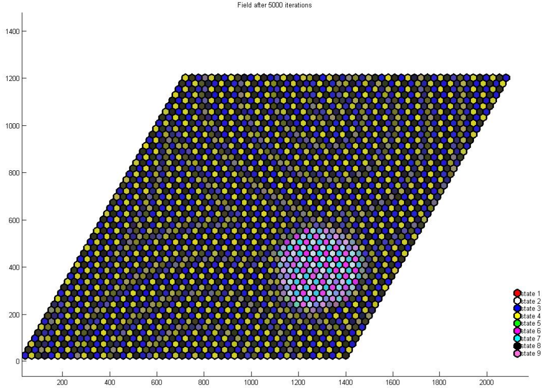
Other 'Evolutionary' Work in Maastricht

Examining the effects of periodic fitness in replicator dynamics

(Trajectory)

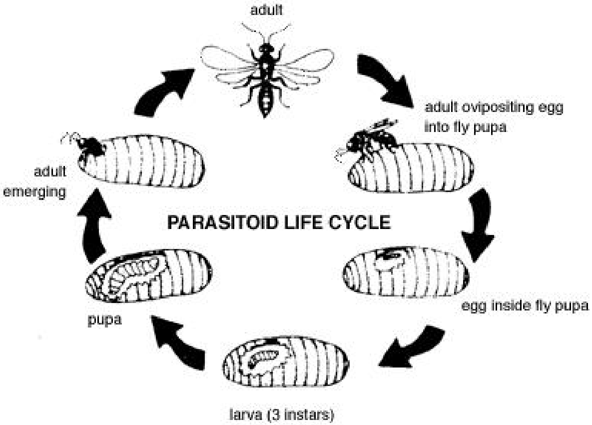
Other ‘Evolutionary’ Work in Maastricht

Examining the effects of local replicator dynamics



Other 'Evolutionary' Work in Maastricht

Studying sex choice ovipositioning behavior of parasitoid wasps



Thanks

Thank you for your attention!
Any comment is welcome!

This presentation will be available at
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