Local Dynamics in Network Formation

Jean Derks, Jeroen Kuipers, Martijn Tennekes*
and Frank Thuijsman

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Maastricht Universitiy, Department of Mathematics

Abstract

We study a dynamic network formation game. Alternately, agents are allowed to add, remove or replace links among them. The goal of each of them is to maximize his own payoff, which is a function of the final network. We focus on local actions, where each agent is allowed to add, remove or replace only one link per turn. We prove that the problem of finding a best global response is $\mathcal{NP}$-hard, while finding a best local response can be done in polynimal time. We show that for a general class of payoff functions, which is based on axiomatic properties, local-Nash and global-Nash networks always exist. Also, we show that the dynamic process of iterated local actions converges with probability 1 to a local-Nash network. Moreover, this local-Nash network is also global-Nash.

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JEL classification: C72, D85

1 Introduction

Social and economic networks play a major role in modern day society. In these networks, individuals are linked if there is a specific relationship between them, for instance a friendship or a trading relationship. Each node in these networks represents an agent (e.g. an individual or an organisation). Via pairwise links agents may benefit from eachother. Consider for instance a network where individuals share valuable information about job openings.

These networks are often formed by the agents themselves rather than by a centralized authority. This implies that the agents determine the structure of the formed networks. In this paper we study the formation of social and economic networks by means of a dynamic model. The literature on models of network formation is emerging, especially in the last decade. Pioneering work has been done by Jackson and Wolinsky (1996) and by Bala and Goyal (2000a). Network formation from a cooperative point of view has been studied

*Corresponding author: martijn.tennekes@micc.unimaas.nl

by Myerson (1977), Slikker and Van den Nouweland (2000), Jackson (2005a) and Slikker et al. (2005), among others. For a brief introduction and overview of literature on models of network formation we refer to Jackson (2005b) and to Van den Nouweland (2005).

Our model is inspired by the one-way flow model that is proposed by Bala and Goyal (2000a) and extended by Galeotti (2006). Bala and Goyal (2000a) model network formation as a non-cooperative game. Here, an agent's action is defined as a set of links. The links of all agents together define a one-way flow network. The links that are formed by agent i are depicted by arcs pointing at i. A payoff function is defined on the formed network as follows. Each agent pays certain costs for each own link, i.e., each link directed at him, and he gains certain profits from being connected to other agents in the formed network. Bala and Goyal (2000a) characterize and prove the existence of Nash networks for games where link costs and profits are homogeneous, i.e. all links are equally expensive and all agents have equal profits. Galeotti (2006) studies heterogeneity among profits and link costs and he characterizes the architecture of Nash network for various settings of heterogeneity. The existence of Nash networks for games with heterogeneous profits and owner-homogeneous link costs, i.e. all links have equal costs with respect to the agent who formes them, is proved independently by Derks et al. (2007) and Billand et al. (2007). A short and elementary proof that is based directly on the same ideas of Billand et al. (2007) is provided by Derks and Tennekes (2008b).

In this paper we extend this line of research in two ways. First, we generalize the one-way flow model by developing a framework of axiomatic properties for the payoff functions. We define these properties in such a way that they are intuitive and that they are sufficient to guarantee the existence of Nash networks. Second, we examine a procedure of local improvements, where agents have four types of elementary actions, which are: passing, adding a link, removing a link, and replacing a link. We will refer to these actions as local actions.

We choose this local approach for several reasons. First of all, local actions are easier to deal with in the analysis of network formation than global actions, which are defined as adding, replacing and deleting multiple links. From a computational point of view, we show that the problem of finding a best global response is $\mathcal{NP}$-hard, while finding a best local response can be done in polynomial time. Second, local actions are more realistic than global ones in applications of social and economic network formation; an individual rather changes one connection at a time, than all at once.

Networks in which no agent can improve by a unilateral local action are called local-Nash networks. Global-Nash networks are defined analogously. We propose a framework of intuitive payoff properties, that guarantees the existence of local-Nash networks. Networks where each agent has at most one outgoing link play a prominent role in our analysis. We prove that local-Nash networks that satisfy this architectural property, are also global-Nash when three payoff properties are satisfied. Thus, our local approach enables us to prove the existence of global-Nash networks given that the payoff function satisfies a set of properties. We show that these payoff properties are independent, and furthermore, we show that all payoff functions studied by Bala and Goyal (2000a) and Galeotti (2006) where link costs are owner-homogeneous satisfy these properties. Moreover, we provide examples of other payoff functions that satisfy these properties. These payoff functions take more aspects of the network architecture into account.

Our dynamic play starts with an initial network that consists of a set of agents and a set of links between them. Alternately, the agents are allowed to perform local actions. The play ends when no agent wants to adjust the network. The goal of each agent is to maximize his payoff, which is a function of the network finally derived. We assume that the agents choose their actions myopically, and that the order in which the agents respond is random.

We consider an iterative procedure of good local responses, where a good local response is a local action for which the payoff of the playing agent does not decrease. These good responses are more realistic in applications of network formation. Especially in large networks, individuals rather prefer ad-hoc to deliberate decision making. We show that this dynamic procedure always converges to a local-Nash network, which is also global-Nash, whenever the payoff function satisfies the axiomatic properties of our framework.

Bala and Goyal (2000a) also propose a model of network formation games on two-way flow networks. The links in these networks are undirected and profits can flow in both directions. Generalizations of this two-way model have been established by Galeotti et al. (2006) and Haller et al. (2007). Many extensions of these models have been studied, like the effect of decay (Bala and Goyal (2000a)), and the effect of link reliability (Bala and Goyal (2000b) and Haller and Sarangi (2005)). Johari et al. (2006) studied a model of traffic routing based on the one-way flow model.

A model that is close to these one-way and two-way flow models of network formation is the connections model introduced by Jackson and Wolinsky (1996). Here, agents form links bilaterally instead of unilaterally. In other words, a link is only formed if both agents choose that link. They study pairwise stable networks, which are equilibrium networks where no agent wishes to delete a link and no pair of agents wishes to form a link. They focus on the tension between pairwise stable networks and efficient networks, i.e., networks with a maximum sum of payoffs. This framework has been extended by Dutta and Muttuswami (1997), Dutta and Jackson (2000), Jackson and Van den Nouweland (2005) and Bloch and Jackson (2006), among others.

Our dynamic process of iterated local actions resembles the one described by Bala and Goyal (2000a). However, they examine only global actions, and multiple agents may play simultaneously during each stage of their model. Watts (2001) and Jackson and Watts (2002) also study dynamic models of network formation, but their model is based on Jackson and Wolinsky (1996).

The outline of this paper is as follows. In Section 2 we present the model and the notations that we use throughout. In Section 3 we study the complexity of determining best global and best local responses. Here, we show that the problem of finding a best global response is $\mathcal{NP}$-hard, while the problem of finding a best local response can be solved in polynomial time. In Section 4 we prove the existence of local-Nash and global-Nash networks for games where the payoff function satisfies a specific set of axiomatic properties. Furthermore, we prove independence of these properties and relate them to payoff functions of the one-way flow model. In Section 5, we study the play of our dynamic game. We provide an iterative procedure of good local responses, and show that it converges to a local Nash network whenever the payoff function satisfies our payoff properties. Finally, in Section 6 we provide concluding remarks.

2 Model and notations

In this section we provide our model of network formation and we introduce the notations that we will use throughout this paper.

2.1 Network

Let N denote a finite set of agents. We define a *network* g on the agent set N as a set of links $g \subseteq N \times N$, where loops are not allowed, i.e. $(i,i) \notin g$ for all $i \in N$. Let $\mathcal{G}$ be the set of all possible networks on N. A *directed path* from i to j in g is a sequence of distinct agents $i_1, i_2, \ldots, i_k$ with, $k \geq 1$, such that $i = i_1$, $j = i_k$ and $(i_s, i_{s+1}) \in g$ for each $s = 1, 2, \ldots, k-1$. Notice that for $k = 1$ we have that $i = i_1$ is a trivial directed path without links from i to himself. An *undirected path* is defined analogously where either (i_s, i_{s+1}) or (i_{s+1}, i_s) is contained in g for each $s = 1, 2, \ldots, k-1$.

For convenience we will use use the symbols '+' and '$-$' for the union respectively the set difference of two networks, or for a network and a single link. These operations are applied from left to right. For instance, the notation $g - g' + (j,i)$ equals $(g \setminus g') \cup \{(j,i)\}$.

Let $\mathrm{Car}(g)$, the carrier of g, denote the set of so-called *active* agents in the network g, being those agents who are begin- or endpoints of a link in g. For a network g we define g^j, the *component* of g that contains agent j, as the network containing all links that are connected to j by some undirected path. Since g^j may be empty, which is the case when j is isolated in g, we assume that j is contained in the carrier of g^j, i.e. let $j \in \mathrm{Car}(g^j)$.

We say that a link (j,i), which is directed to i, is *owned* by i. Let g_{-i} denote the network obtained from g after removing the links owned by i. Notice that an outgoing link of i, e.g. (i,j), may still exist in g_{-i}. Further, we define $g_{-ij} = g^j_{-i} + (j,i)$, where g^j_{-i} means $(g_{-i})^j$, i.e. the component of g_{-i} with j being active. We will come back to this definition and the related definition *beneficiality* in Section 4.

For each agent i, let $\pi_i : \mathcal{G} \to \mathbb{R}$ be a payoff function. In Section 4 we introduce axiomatic properties for payoff functions in general. These properties are based on the payoff functions of the one-way flow model studied by Bala and Goyal (2000a) and generalized by Galeotti (2006). We will refer to such specific payoff functions as B&G functions. Before defining the B&G functions, we need the following notation. Let $N_i(g) = \{j \in N : \text{ a directed path from } j \text{ to } i \text{ exists in } g\}$

be the set of agents who are *observed* by i in g, and let $N_i^d(g) = \{j : (j, i) \in g\}$ be the set of *neighbors* of i in g. Note that $i \in N_i(g)$ and $i \notin N_i^d(g)$.

Payoff function $\pi_i(g)$ is called a B&G function if

$$\pi_i(g) = \sum_{j \in N_i(g)} v_{ij} - \sum_{j \in N_i^d(g)} c_{ij} \tag{1}$$

where v_{ij} is the profit that agent i receives from being connected to j and c_{ij} is the cost of link (j, i) for agent i. Let all profits and costs be non-negative. Without loss of generality we may assume that $v_{ii} = 0$ for each $i \in N$. We say that link costs are *homogeneous* if there is a constant c with $c_{ij} = c$ for all $i, j \in N$, $i \neq j$. We say that link costs are *owner-homogeneous* if for each agent i there is a constant c_i with $c_{ij} = c_i$ for all $j \in N \setminus \{i\}$. Otherwise, link costs are *heterogeneous.* These definitions also apply to the profits.

2.2 Network formation game

Given a set of agents N and a payoff function π_i for each agent i, a network formation game proceeds in stages $1, 2, 3, \ldots$. Let g_t be the network at the beginning of stage t, which is known to all agents. The initial network g_1 can be any network in $\mathcal{G}$. Then, at stage t according to a probability device, an agent, say i, is selected. We assume that at each stage all agents have positive (stage independent) probabilities of being chosen. Now, stage t proceeds by allowing agent i to modify the network g_t by adjusting his set of links. Thus, a new network g_{t+1} results, which marks the start of stage $t+1$. The game ends with network g_* if no agent wants to adjust his links. We assume that during the play, the agents are myopic. As for the stage adjustments we examine two cases: one of local adjustments and one of global adjustments. In the first case the actions of player i are restricted to (1) passing, (2) adding a new link pointing at i, (3) deleting a link pointing at i, or (4) a replacement, which is a combination of the previous two. These four types of actions are called *local actions.* In the second case player i is allowed to completely change the set of links pointing at him. These actions are called *global actions.*

Formally, we define an action of agent i as a set of agents, denoted as $S_i \subseteq N \setminus \{i\}$. For a global action, there are no restrictions on S_i. For a local action we require $|N_i^d(g) \setminus S_i| \leq 1$ and $|S_i \setminus N_i^d(g)| \leq 1$. The network, after i chooses to link with the agents in S_i, is described by

$$g_{-i} + \{(j, i) : j \in S_i\}.$$

A local action S_i of agent i is called a *good local response* if

$$\pi_i\big(g_{-i} + \{(j, i) : j \in S_i\}\big) \geq \pi_i(g).$$

A local action S_i of agent i is called a *best local response* if

$$\pi_i\big(g_{-i} + \{(j, i) : j \in S_i\}\big) \geq \pi_i\big(g_{-i} + \{(j, i) : j \in T_i\}\big),$$

for all local actions T_i. A network g is called a *local-Nash network* if $N_i^d(g)$ is a best local response for all $i \in N$. A network g is called a *strict local-Nash network* if $N_i^d(g)$ is the unique best local response for all $i \in N$. Analogous definitions apply for the global case.

The following example with three agents illustrates how the dynamic formation game is played.

Example 1 *Let the set of agents be $N = \{1,2,3\}$ and let π be a B&G function as described in (1), where $v_{ij} = 2$ and $c_{ij} = 1$ for all agents i and j.*

Table 1: Play of the game.

Stage	*Chosen agent*	*Local action*	*Obtained network*
1	1	add $(3,1)$	$g_2 = \{(3,1)\}$
2	3	add $(1,3)$	$g_3 = \{(3,1),(1,3)\}$
3	2	add $(3,2)$	$g_4 = \{(3,1),(1,3),(3,2)\}$
4	1	replace $(3,1)$ by $(2,1)$	$g_5 = \{(2,1),(1,3),(3,2)\}$

Let the initial network in this example, g_1, be the empty network and let the agents play local actions. The play of the game is shown in Table 1. In the second column the chosen agent is given, but the agents do not know the order in which they are chosen in advance. The corresponding networks are depicted in Figure 1. Notice that all played local actions in this example are best local responses. Furthermore, the last network, g_5, is strict local-Nash.

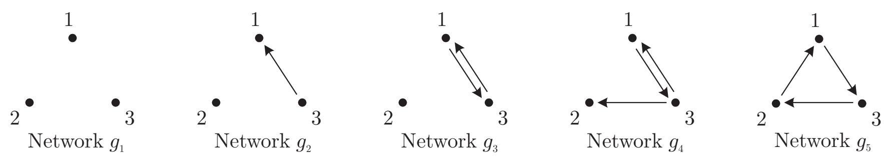

Figure 1: The networks obtained in Example 1

3 Best response

In this section we analyse the complexity of finding both best local and best global responses. Obviously, there are 2^{n-1} possible global actions that an agent can perform, where n is the number of agents. In the following theorem we show that the number of local actions that an agent can perform, is bounded by the square of n.

Theorem 2 *The number of possible local actions that an agent can perform, is bounded by n^2, where n is the number of agents.*

Proof. The number of possible local actions that agent i can perform, depends on the number of neighbors of i in g. If we denote this number by $m \leq n-1$, then agent i can do $n-m-1$ additions, m deletions and $(n-m-1)m$ replacements. Hence, the number of possible local actions for agent i equals

$$\begin{aligned} n-m-1+m+(n-m-1)m &= (n-1)+(n-m-1)m \\ &\leq (n-1)+(n-m-1)(n-1) \\ &= (n-1)(n-m) \\ &\leq n^2. \end{aligned}$$

□

This implies that finding a best local response can be done within polynomial time. The problem of finding a best global response is often difficult to solve. To this end, we restrict to a relatively small class of payoff functions, namely B&G functions with homogeneous link costs. For this class, we show in the following theorem that the *Best Global Response Problem* (*BGRP* in short) is $\mathcal{NP}$-hard.

Theorem 3 *Let π be a B&G function. Then BGRP is $\mathcal{NP}$-hard, even when link costs are homogeneous.*

Proof. We prove this by reduction from the Minimum Set Cover problem (MSC), which is a well-known $\mathcal{NP}$-hard problem (see Karp (1972)). Let $\mathcal{K} = \{K_1, K_2, \ldots, K_k\}$ be a collection of k subsets of a finite set $X = \{1, 2, \ldots, x\}$ such that $X \subseteq \bigcup_{j=1}^{k} K_j$. MSC is the problem of finding a subset $\mathcal{K}' \subseteq \mathcal{K}$ of minimum cardinality such that every element in X belongs to at least one member of $\mathcal{K}'$. W.l.o.g. such a set cover $\mathcal{K}'$ may be assumed to exist.

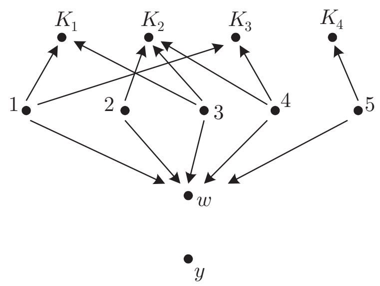

Figure 2: Network g

Now we show how to reduce MSC to BGRP. Let the agent set be initiated as $N = \{1, \ldots, x, K_1, K_2, \ldots, K_k, w, y\}$. Let $g \in \mathcal{G}$ be a network on N, built up as follows: for each agent $i \in X$ we create a link (i, w) and for each agent $i \in K_j$ we create a link (i, K_j). In Figure 2 an example of such a network is shown. Let the profits of agent y have the following values:

$$v_{yi} = \begin{cases} 1 & \text{if } i \in X; \\ 1 - \frac{1}{2k} & \text{if } i \in \mathcal{K}; \\ 0 & \text{if } i \in \{w, y\}. \end{cases}$$

Let the link costs be homogeneous; let $c = 1$.

We show that the problem of finding a best global response S_y for agent y with respect to g is equivalent to the problem of finding a minimum subset of $\mathcal{K}$ that covers X.

Observe that we may restrict to $S_y \subseteq \{K_1, \ldots, K_k, w\}$, because every $i \in X$ is an element of some K_j, and therefore agent y would receive at least as much payoff from replacing i by K_j. Also, if $w \in S_y$, then $S_y = \{w\}$, since the cost of any additional link exceeds the extra profits. Hence, either $S_y = \{w\}$ or $S_y \subseteq \mathcal{K}$. Observe that the action $\{w\}$ yields the payoff $x - 1$ for agent y.

Let $\mathcal{K}' \subseteq \mathcal{K}$ and let T_y be an action defined as $T_y = \mathcal{K}'$. Then T_y yields the following payoff for agent y:

$$\pi_y(g_{-y} + \{(i, y) : i \in T_y\}) = k'(1 - \frac{1}{2k}) + t - k' = \frac{-k'}{2k} + t$$

where $k' = |\mathcal{K}'|$ and t is the number of members of X that are covered by $\mathcal{K}'$.

If $\mathcal{K}'$ does not cover X, then $t \leq x-1$, and hence $\frac{-k'}{2k} + t \leq \frac{-k'}{2k} + x - 1 < x - 1$. In other words, the action T_y yields a payoff which is strictly less than the payoff $x - 1$ which corresponds to the action $\{w\}$. Hence we conclude that if $\mathcal{K}'$ does not cover X, then the corresponding action T_y is not a best global response.

If $\mathcal{K}' \subseteq \mathcal{K}$ covers X, then $t = x$ and hence $\frac{-k'}{2k} + t = \frac{-k'}{2k} + x > x - 1$. Thus, the action T_y yields a strictly higher payoff than the payoff $x - 1$ which corresponds to the action $\{w\}$. So every action that is a set cover yields a strictly higher payoff than the payoff from the action $\{w\}$. Of all actions that are set covers, the ones with the lowest cardinality are best global responses, because the payoff $\frac{-k'}{2k} + x$ is maximal if k' is minimal. We therefore conclude that each best global response of agent y with respect to network g is defined as $S_y = \mathcal{K}'$ where $\mathcal{K}'$ is a minimum set cover.

Since the transformation from any MSC instance to an BGRP instance can be done in polynomial time and since MSC is $\mathcal{NP}$-hard (see Karp (1972)), it follows that BGRP is also $\mathcal{NP}$-hard. □

Observe that the BGRP can be interpreted as the problem of maximizing a set function, since the playing agent i and the network g_{-i} are fixed in the BGRP. Hence we define a specific set function $f : 2^{N \setminus \{i\}} \to \mathbb{R}$ as

$$f(S) = \pi_i(g_{-i} + \{(j, i) : j \in S\})$$

for each $S \subseteq N \setminus \{i\}$, where network g_{-i} and agent i are fixed.

Set function f is called *submodular* if

$$f(S) + f(T) \geq f(S \cup T) + f(S \cap T) \text{ for all } S, T \subseteq N \setminus \{i\},$$

and *supermodular* if the left-hand side is less than or equal to the right-hand side.

For maximizing supermodular set functions in general, which is equivalent to minimizing submodular set functions, Grötschel et al. (1981) proposed a polynomial-time algorithm. Alternative algorithms that are more efficient in practice are proposed independently by Schrijver (2000) and Iwata et al. (2001).

Garey and Johnson (1979) show that the problem of maximizing submodular set functions is $\mathcal{NP}$-hard, due to the fact that it is a general case of the max-cut problem. The problem of maximizing submodular set functions has also been studied by Nemhauser et al. (1978), Lovasz (1983) and Lee et al. (1996), among others.

It can be shown that f is submodular whenever the corresponding payoff function π is a B&G function, due to the fact that for disjoint actions S and T, the sets $N_i(g_{-i} + \{(j,i) : j \in S\})$ and $N_i(g_{-i} + \{(j,i) : j \in T\})$ may intersect. Hence, the BGRP is a special case of the $\mathcal{NP}$-hard problem of maximizing a submodular set function. In Theorem 3 we have shown that this special case is also $\mathcal{NP}$-hard, even for B&G functions with homogeneous link costs.

4 Nash networks

In this section we study the existence of Nash networks. Bala and Goyal (2000a) show that global-Nash networks exist when payoff functions are B&G functions with homogeneous link costs and profits. For B&G functions with owner-homogeneous link costs, i.e. $c_{ij} = c_i$, and heterogeneous profits, the existence of Nash networks has been proved independently by Derks et al. (2007) and Billand et al. (2007).[1] By means of a counterexample, Derks et al. (2007) show that Nash networks may fail to exist for the heterogeneous link costs case, even if link costs are arbitrarily 'close' to the situation of owner-homogeneity, i.e. $|c_{ij} - c_{ik}| \leq \epsilon$, for an arbitrarily small $\epsilon > 0$, and for all i, j, and k.

In this section we prove that global-Nash networks exist for a class of payoff functions that is defined by a framework of axiomatic payoff properties. In Subsection 4.1, we introduce three of these properties. We prove that local-Nash networks with some specific architecture are also global-Nash when the payoff function satisfies these properties. Then, in Subsection 4.2, we show the existence of local-Nash networks when the payoff function satisfies a specified set of properties. We choose the properties in such a way that they are intuitive and such that they allow us to proceed with our line of proof in the most general way. In Subsection 4.3, we show that the properties are independent from each other, and in Subsection 4.4, we show which B&G functions satisfy the properties, and furthermore, we provide examples of non B&G functions that satisfy the properties as well.

[1] Derks and Tennekes (2008b) provide an alternative and easy accessible proof that is based directly on the same ideas of Billand et al. (2007).

4.1 Local-Nash and global-Nash networks

Let a network be called *proper* if the outdegree of each agent is at most one. An illustrative proper network is depicted in Figure 3.

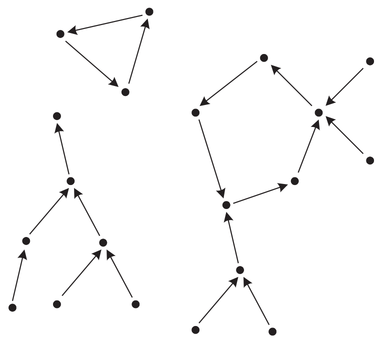

Figure 3: A proper network

We show that for a specific class of payoff functions every proper local-Nash network is also global-Nash. This class consists of all payoff functions that have three properties which we will define next: **DA** (short for disjoint additivity), **NA** (short for naturality), and **DE** (short for downstream efficiency).

Two networks g and g' are said to be *disjoint* with respect to an agent i, or i-disjoint, if no agent or only agent i is active in both g and g': $\mathrm{Car}(g) \cap \mathrm{Car}(g') \subseteq \{i\}$.

Property DA We say that a payoff function π is *disjoint additive* (**DA** for short), if for each two networks g and g', disjoint w.r.t. an agent i, we have

$$\pi_i(g + g') = \pi_i(g) + \pi_i(g').$$

Recall that $g_{-ij} = g^j_{-i} + (j, i)$, where g^j_{-i} is the component in g_{-i} where agent j is active. The payoff with repect to network g_{-ij} is an indication for the importance of the link (j, i) for agent i in the network g. Notice that link (j, i) does not have to be contained in g. We say that link (j, i) is *beneficial* in g if $\pi_i(g_{-ij}) \geq \pi_i(g_{-i})$. A network is called *beneficial* if all links in that network are beneficial.

In the sequel, we assume that a payoff function π_i obeys $\pi_i(g) = 0$ whenever agent i has no incoming links in g, i.e. $g = g_{-i}$. Observe that this also applies to B&G functions, since we assumed w.l.o.g. that $v_{ii} = 0$. Therefore, we assume that link (j, i) is beneficial in g if $\pi_i(g_{-ij}) \geq 0$.

Lemma 4 *If network g is proper, and π is disjoint additive, then*

$$\pi_i(g) = \sum_{j \in N_i^d(g)} \pi_i(g_{-ij}) \tag{2}$$

for each agent i.

Proof. Let g be a proper network. We claim that g_{-ij} and g_{-ik} are i-disjoint whenever $j, k \in N_i^d(g)$, $j \neq k$. This is true whenever $\mathrm{Car}(g_{-i}^j) \cap \mathrm{Car}(g_{-i}^k) = \emptyset$.

Suppose to the contrary that an agent r exists such that $r \in \mathrm{Car}(g_{-i}^j) \cap \mathrm{Car}(g_{-i}^k)$. Hence an undirected path exists between j and k in g_{-i} that goes through r. (Notice that this path does not visit agent i, because in network g_{-i}, agent i is adjacent to at most one link.) Since g is proper and agents j and k both have an outgoing link to i in network g, they cannot have other outgoing links in g. Hence j and k have no outgoing links in g_{-i}. Thus, the undirected path between j and k through r starts with a link directed to j and ends with a link directed to k. Hence, at least one agent on this path has two outgoing links, which contradicts that g is proper. We conclude that g_{-ij} and g_{-ik} are i-disjoint.

Let network g' be defined as

$$g' = g - \sum_{j \in N_i^d(g)} g_{-ij}.$$

Notice that i does not have any incoming links in g'. Therefore, $\pi_i(g') = 0$. Obviously, g' is disjoint from g_{-ij} in network g, for each $j \in N_i^d(g)$.

Since $\pi_i(g') = 0$ and $g = g' + \sum_{j \in N_i^d(g)} g_{-ij}$, where all g_{-ij} and g' are i-disjoint, we obtain (2) by **DA**. □

The next payoff property states that connecting to an agent who is already observed is not a strictly improving action.

Property NA We say that π is *natural* (**NA** for short) if

$$\pi_i(g + (k, i)) \leq \pi_i(g)$$

whenever $k \in N_i(g)$, i.e. there is a directed path from k to i in the network g.

Thus, in a network where i already observes k via another link, say (j, i), the addition of (k, i) is not an improving action due to **NA**. The next payoff property can be seen as a "twin" property.

Property DE Payoff function π satisfies **DE** (short for *downstream efficiency*) if

$$\pi_i(g + (k, i)) \leq \pi_i(g + (j, i))$$

for any network g where $(j, i) \notin g$ and $(k, i) \notin g$ and where a directed path exists from k to j in g_{-i}.

Due to **DE**, the addition of link (j, i) is at least as good as the addition of (k, i). Observe that the difference between **NA** and **DE** is that in the situation where **NA** is applicable, link (j, i) does exist (on the directed path from k to i), whereas in the situation where **DE** is applicable, link (j, i) does not exist.

In the following theorem we show that proper local-Nash networks are also global-Nash whenever the payoff function satisfies the three introduced properties. Here, we need the following definition. Let agent i be a *topagent* in network

g whenever he observes all agents in his component, i.e. $N_i(g) = \mathrm{Car}(g^i)$. Notice that a topagent either is contained in a cycle, or he has an outdegree of 0.

Theorem 5 *Let the payoff function π satisfy **DA**, **NA** and **DE**. Then a proper local-Nash network is global-Nash.*

Proof. Let g be proper and local-Nash. Suppose that g is not global-Nash, and let i be the agent who can strictly improve by playing a best global response $\tilde{S}$. Let $S = N_i^d(g)$.

Suppose that g_{-i} is a component. Let $T(g) \subseteq \mathrm{Car}(g)$ be the set of topagents in g. Since g is proper, at least one topagent is contained in $\mathrm{Car}(g)$, i.e. $T(g) \neq \emptyset$. Since g_{-i} is a component, a directed path exists from k to j in g_{-i}, for all $k \in \mathrm{Car}(g)$ and $j \in T(g)$.

Suppose that $|S| \geq 2$. Let j and k be two agents in S. Since g_{-i} is a component, an undirected path exists between j and k in g_{-i}. Therefore, an undirected cycle with (j, i) and (k, i) exists in g. Since this cycle is not directed, an agent on it exists who has at least two outgoing links. Hence we have derived a contradiction with properness of g. Thus we conclude that $|S| \leq 1$.

Suppose that an agent $k \in \tilde{S} \setminus T(g)$ exists. If $\tilde{S} \cap T(g) \neq \emptyset$, then by **NA** it follows that $\tilde{S} - k$ is at least as good as $\tilde{S}$. If $\tilde{S} \cap T(g) = \emptyset$, then by **DE** the action $\tilde{S} - k + j$ is at least as good as $\tilde{S}$, with $j \in T(g)$. Hence we may assume that $\tilde{S} \subseteq T(g)$.

Suppose that $|\tilde{S}| \geq 2$. Let j and k be two agents in $\tilde{S}$. Since both agents are topagents, a directed path exists from k to j (and vice versa) in g_{-i}, and therefore, by **NA** it follows that $\tilde{S} - k$ is at least as good as $\tilde{S}$. Hence, we may assume that $|\tilde{S}| \leq 1$.

Since $\tilde{S}$ is an improving action of agent i, we have $\tilde{S} \neq S$. Since $|S| \leq 1$ and $|\tilde{S}| \leq 1$, we conclude that $\tilde{S}$ is a local action. Hence we have derived a contradiction, and therefore we conclude that g is global-Nash.

Now suppose that g_{-i} contains multiple components. Then, for each component g'_{-i} in g_{-i}, the previous analysis can be applied. Thus, the sets S and $\tilde{S}$ with respect to g'_{-i} both have cardinality 1. By **DA** and Lemma 1, changing from S to $\tilde{S}$ with respect to g'_{-i} is also a strictly improving action with respect to g_{-i}. Since this action is a local action, we have derived a contradiction. Hence, we conclude that g is global-Nash. □

For more general payoff functions, proper local-Nash networks need not to be global Nash as can be seen by the following example.

Example 6 *Consider the following payoff function:*

$$\pi_i(g) = \begin{cases} |L(g)| - 1 & \text{if } N_i^d(g) = L(g) \neq \emptyset; \\ 0 & \text{otherwise,} \end{cases} \tag{3}$$

*where $L(g)$ is the set of agents who do not have incoming links in g. It can be easily verified that this payoff function satisfies **DA**. It also satisfied **NA**, because $\pi_i(g) = 0$ whenever a directed path from k to i (containing multiple links) exists in g. However, it does not satisfy **DE**.*

Consider a network on 4 agents. Let the payoff functions $\pi_1(g)$ and $\pi_3(g)$ be defined as (3), and let $\pi_2(g) = \pi_4(g) = 0$ for all networks g. Then, the network depicted in Figure 4(a) is a proper local-Nash network. However, it is not global-Nash, because agent 1 can switch to the network depicted in Figure 4(b), which yields payoff 1 instead of 0. This network is global-Nash.

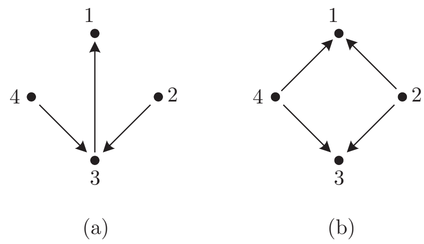

Figure 4: A proper local-Nash network that is not global-Nash (a), and a global-Nash network (b).

In the next subsection, we prove the existence of proper local-Nash networks. By Theorem 5 we know, that these networks are also global-Nash.

4.2 Existence of local-Nash networks

To prove the existence of local-Nash networks, we introduce four new properties. These properties only regard beneficiality. Recall that a link (j, i) is beneficial in g whenever $\pi_i(g_{-ij}) \geq 0$.

Property BT Payoff function π satisfies **BT** (short for *beneficial topagent*) if the following holds. Let link (k, i) be beneficial in network g, and suppose there are topagents in the component g^k_{-i}. Then there is a topagent j in g^k_{-i} such that $\pi_i(g_{-ij}) \geq \pi_i(g_{-ik})$.

Notice that this property is implied by **DE**. It compares the beneficiality of two links, (j, i) and (k, i). The next three properties that we introduce concern how beneficiality of a link is preserved when the network is changed, or how beneficiality of a link depends on the beneficiality of other links.

Property BF Payoff function π satisfies **BF** (short for *beneficial farthest*) if the following holds. Let link (k, i) be beneficial in network g; let the component g^k_{-i} be proper and let agent i be active in g^k_{-i} (there is an outgoing link at i in g^k_{-i}). Then also link (j, i) is beneficial where j is the agent farthest away from i (counted in number of links) in network g.

Notice that since component g^k_{-i} is proper and i is active in it, agent j is the (unique) topagent who is farthest away from i in network g. Property **BF** is also implied by **DE**. However, it is independent from **BT** as we will see in Subsection 4.3. Furthermore, **BT** and **BF** do not imply **DE**. This is illustrated by Example 6, where the payoff function given by (3) does not satisfy **DE**, whereas it satisfies both **BT** and **BF**, since $\pi_i(g_{-ij}) = 0$ for each network g and each agent j.

The following property describes that beneficial links remain beneficial while the network grows:

Property BG Payoff function π satisfies **BG** (short for *beneficial growth*) if $\pi_i((g + (k, r))_{-ij}) \geq 0$ for any two agents $k,\ r$, whenever $\pi_i(g_{-ij}) \geq 0$.

Notice that we may have $r = i$. In that case, **BG** states that link (j, i) remains beneficial when i adds link (k, i).

The final property states that beneficiality is preserved when we delete a spoke from the network. Here, a *spoke* in a network g is a link (k, r) such that both agents k and r reside on a directed cycle, with link (k, r) not being part of it.

Property BS Payoff function π satisfies **BS** (*beneficial shrink*) if $\pi_i((g - (k, r))_{-ij}) \geq 0$ whenever $\pi_i(g_{-ij}) \geq 0$ and link (k, r) is a spoke in g.

The properties **BF**, **BG** and **BS** are trivially satisfied by non-negative valued payoff functions. An example of such a function is $\pi_i(g) = |N_i(g) \setminus \{i\}|$ being the number of agents in g observed by i. This function also satisfies **DA**, **NA** and **BT**.

Let us call a payoff function *orderly* if it satisfies the properties **DA**, **NA**, **BT**, **BF**, **BG**, and **BS**.

Let $\kappa(g)$ be the *connection number* of network g, and define it as

$$\kappa(g) = \sum_{i \in N} |N_i(g)|.$$

Observe that the addition of a link or the deletion of a spoke does not decrease the connection number. Now, we state our main result:

Theorem 7 *For orderly payoff functions any proper, beneficial network with maximal connection number is a local-Nash network.*

Proof. Let g be a proper, beneficial network such that among these networks the connection number $\kappa(g)$ is maximal. Observe that the empty network is proper and beneficial. We prove that g is local-Nash by deriving a contradiction in the sense that otherwise a proper, beneficial network exists with a higher κ-value than g.

Suppose there is a local action by agent i that strictly increases i's payoff. Clearly, this action is not a pass. This local action is neither a deletion because all links of agent i are beneficial, and (2) applies due to g being proper and π_i satisfying **DA**.

Suppose the strictly improving local action is a replacement, say link (k, i) is replaced by link (j, i), and let the obtained network be $g' = g - (k, i) + (j, i)$. Notice that k is the unique topagent in g^k_{-i}, since g is proper. If both agents k and j are in component g^k_{-i}, then, by property **BT** it follows that $\pi_i(g_{-ij}) \leq \pi_i(g_{-ik})$. However, $g - g_{-ik}$ and $(g')_{-ij} = g_{-ij}$ are i-disjoint and their union is g', so that by **DA** we have

$$\pi_i(g - g_{-ik}) + \pi_i(g_{-ik}) = \pi_i(g) < \pi_i(g') = \pi_i(g - g_{-ik}) + \pi_i(g_{-ij}),$$

i.e., $\pi_i(g_{-ij}) > \pi_i(g_{-ik})$, a contradiction.

Therefore, agents k and j are in different components of g_{-i}. The networks g_{-ik} and $g - g_{-ik} + (j, i)$ are i-disjoint, with union equal to $g + (j, i)$, and the networks g^k_{-i} and $g - g_{-ik} + (j, i)$ are i-disjoint with union $g - (k, i) + (j, i)$. Applying **DA** twice, we obtain

$$\begin{array}{rcl} \pi_i(g + (j, i)) & = & \pi_i(g - g_{-ik} + (j, i)) + \pi_i(g_{-ik}) \\ & \geq & \pi_i(g - g_{-ik} + (j, i)) + \pi_i(g^k_{-i}) \\ & = & \pi_i(g - (k, i) + (j, i)). \end{array}$$

The inequality follows, since $\pi_i(g^k_{-i}) = 0$, and since $\pi_i(g_{-ik}) \geq 0$, by the beneficiality of (k, i). We conclude that the addition of (j, i) is at least as good as the replacement of link (k, i) in g by (j, i).

So, we may assume that the strict improving local action is an addition. Let this addition be (j, i) and let the obtained network be

$$g' = g + (j, i). \tag{4}$$

If the component $(g')^j_{-i}$, which is equal to g^j_{-i}, is already linked up with i, say $(k, i) \in g$ and $k \in \mathrm{Car}(g^j_{-i})$, then k is the unique topagent in g^j_{-i}, due to the properness of g. So, there is a directed path from j to k in g, and with (k, i) there is a directed path from j to i in g, implying $\pi_i(g') = \pi_i(g + (j, i)) \leq \pi_i(g)$ because of **NA**. This is a contradiction to the fact that adding (j, i) is strictly improving.

Therefore, the component $(g')^j_{-i} = g^j_{-i}$ is not linked up with i in g, and by **BT** we may assume that j is a topagent in g^j_{-i}.

Due to **BG**, the network g' is beneficial. Also, the number $\kappa(g')$ is higher than $\kappa(g)$, so that g' cannot be proper, because by assumption there cannot be proper and beneficial networks with connection number higher than $\kappa(g)$. The only outdegree changed by going from g to g' is the one of agent j. Therefore, the outdegree of j in g' equals 2, say next to link (j, i), also (j, k) is present in g', for a $k \in \mathrm{Car}(g^j_{-i})$. Since j is a topagent in g^j_{-i} there is a directed path from k to j in g^j_{-i} and this is also a directed path in $(g')^j_{-k} = g^j_{-k} + (j, i)$.

Extending the directed path from k to j via (j, i) in $(g')^j_{-k}$ in a unique way (since it is a proper network), we arrive at an agent, say r, farthest away from

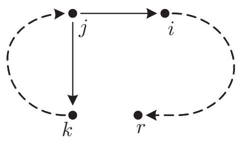

Figure 5: The addressed links and directed paths (dashed arcs) of network g'.

k (see Figure 5). Since (j, k) is beneficial for k, in g', also (r, k) is beneficial in g' due to **BF**.

Consider the addition of (r, k) in g'. From **BG** and (r, k) being beneficial in g', we conclude that $g' + (r, k)$ is beneficial. Further, (j, k) is a spoke in this network. After deletion of this spoke, by **BS** we again obtain a beneficial network

$$g'' = g' + (r, k) - (j, k), \tag{5}$$

with a connection number at least as high as $\kappa(g')$ and thus higher than $\kappa(g)$. Hence g'' cannot be proper. This implies that the outdegree of agent r is greater than 1 in g''. Besides (r, k) we have another link, say (r, s), and s is necessarily located on the unique directed path from k to r in g', for otherwise, s would be farther away from k than r is.

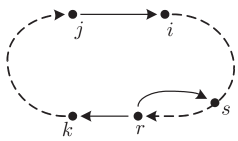

Figure 6: Network g'', where link (r, s) is a spoke.

This directed path still exists in g'', and together with (r, k) it forms a cycle in g'' with (r, s) being a spoke of it (see Figure 6). By deletion of (r, s) we obtain a beneficial network

$$g''' = g'' - (r, s), \tag{6}$$

due to **BS**. Its connection number is higher than the one of g, and it is proper. This is a contradiction by our assumption that g is a proper network with maximal connection number. We conclude that there are no strictly improving additions available, i.e., g is local-Nash. $\square$

Proper, beneficial networks with maximal connection numbers are not the only local-Nash networks. The following example shows that even among the non-proper networks local-Nash networks may be found.

Example 8 *Consider the B&G function $\pi_i(g) = |N_i(g)| - |N_i^d(g)|$. This payoff function is orderly as we will see in Subsection 4.4. The network depicted in Figure 7 is local-Nash, but not proper since agent i has two outgoing links.*

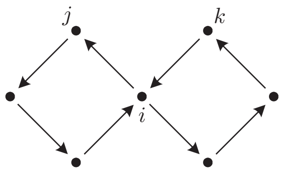

Figure 7: A local-Nash network that is not proper.

Notice that this network is not strict local-Nash, because the replacement of (j, i) *by* (j, k) *yields the same payoff for agent* j*. When agent* i *subsequently removes the spoke* (k, i)*, we obtain a local-Nash network, which is also proper.*

When we relate the previous theorem with Theorem 5 (saying that proper local-Nash networks are also global-Nash if the payoff function satisfies **DA**, **NA** and **DE**), we obtain the following corollary.

Corollary 9 *For any payoff function that satisfies* ***DA****,* ***NA****,* ***DE****,* ***BG*** *and* ***BS****, global-Nash networks exist. Specifically, the proper and beneficial networks with maximal connection number are global-Nash.*

Proof. Since **DE** implies **BT** and **BF**, the payoff function is orderly. By theorem 7 we know that proper local-Nash networks exist. By Theorem 5 we know that these networks are also global-Nash. □

4.3 Property independence

In this subsection, we show the independence of the six properties that define orderliness. This is done by an exposition of examples of payoff functions, fulfilling all but one property.

Theorem 10 *The properties* ***DA****,* ***NA****,* ***BT****,* ***BF****,* ***BG****,* ***BS*** *are independent of each other.*

Proof. We show that for each property a payoff function exists which does not satisfy that property while it does satisfy all other properties.

The following payoff function satisfies all properties, except **DA**:

$$\pi_i(g) = |N_i(g) \setminus \{i\}|^2 \tag{7}$$

Property **DA** is not satisfied, because for any two i-disjoint networks g and g' we have $|N_i(g) \setminus \{i\}|^2 + |N_i(g') \setminus \{i\}|^2 < |N_i(g \cup g') \setminus \{i\}|^2$. The properties **NA** and **BT** are trivially satisfied and the others because of non-negativity, i.e. $\pi_i(g) \geq 0$ for all networks g.

The following payoff function satisfies all properties, except **NA**:

$$\pi_i(g) = |N_i^d(g)| \tag{8}$$

Property **NA** is not satisfied, because $\pi_i(g + (k,i)) > \pi_i(g)$ for a network g where $(k,i) \notin g$, and where a directed path from k to i exists. Property **DA** is clearly satisfied, and also the four properties that concern beneficiality, because $\pi_i(g_{-ij}) = 1$ for all g and j.

The following payoff function satisfies all properties, except **BT**:

$$\pi_i(g) = \sum_{\text{component } g' \subseteq g} \begin{cases} 1 & \text{if } N_i^d(g') = \{k\}, \text{where } N_k^d(g') = \emptyset; \\ 0 & \text{otherwise.} \end{cases}$$

This payoff function does not satisfy **BT**, because for network $g = \{(k,j)\}$, we have $\pi_i(g_{-ij}) < \pi_i(g_{-ik})$. Property **NA** is trivially satisfied because $\pi_i(g) = 0$ for each network g where agent i has at least two incoming links from the same component. Property **DA** is satisfied, since the payoff function is built up componentwise. The remaining properties are trivially satisfied since all payoffs are non-negative.

The following payoff function satisfies all properties, except **BF**.

$$\pi_i(g) = \begin{cases} 0 & \text{if } N_i(g) = N, 1 \notin N_i^d(g), |N_i^d(g)| = 1; \\ -|N_i^d(g)| & \text{otherwise.} \end{cases}$$

In this payoff function, agent 1 is a special member of N. Property **BF** is not satisfied, because of the following. Let g be a network where all agents in $N \setminus \{i\}$ are contained in one directed cycle, that includes link (i,k). Further, let 1 be the agent who is farthest away from i. Then $\pi_i(g_{-ik}) = 0$ while $\pi_i(g_{-i1}) = -1$.

Property **DA** is satisfied because the set $N_i^d(g)$ is built up componentwise.

Property **NA** is satisfied, because adding a link (k,i) to network g where a directed path from k to i exists, implies that i will have multiple links, and therefore his payoff will decrease.

Property **BT** is satisfied because a link (k,i) can only be beneficial if $N_i(g_{-ik}) = N$, which implies that k is a topagent.

Property **BG** is satisfied because of the following. A link (j,i) can only be beneficial in g if $N_i(g_{-ij}) = N$. Therefore, $N_i((g + (k,r))_{-ij}) = N$, for any link (k,r). Further, $|N_i^d((g + (k,r))_{-ij})| = 1$. Hence we conclude that link (j,i) is also beneficial in $g + (k,r)$.

By the same reasoning, property **BS** is satisfied.

The following payoff function satisfies all properties, except **BG**.

$$\pi_i(g) = \left|N_i^d(g) \cap T(g_{-i})\right| - \left|N_i^d(g) \setminus T(g_{-i})\right| \tag{9}$$

where $T(g_{-i})$ is the set of topagents who do not have outgoing links in g_{-i}.

This payoff function does not satisfy **BG**, because due to the addition of link (k, r) to g, the set T may shrink, and thus, $\pi_i(g)$ may decrease. Property **DA** is satisfied, because the sets $N_i^d(g)$ and $T(g)$ are built up componentwise. Property **NA** is satisfied, because in a network g where a directed path exists from k to i, we have $k \notin T(g)$, and therefore the payoff does not increase when (k, i) is added.

Property **BT** is trivially satisfied since (k, i) can only be beneficial if k is a topagent in g_{-i}, by (9). Furthermore, a topagent k in g_{-i} such that (k, i) is beneficial, has no outgoing links in g_{-i} (since $k \in T(g_{-i})$). Therefore, if g is proper, and if i has an outgoing link to the component g^k_{-i}, then it follows that k is the agent who is farthest away from i in g. Hence **BF** is also satisfied.

Property **BS** is satisfied because the deletion of a spoke (k, r) in g does neither affect the set $T(g_{-i})$ nor the set $N_i^d(g)$.

The following payoff function satisfies all properties, except **BS**.

$$\pi_i(g) = \left|K_i(g)\right| - \left|N_i^d(g)\right| \tag{10}$$

where $K_i(g)$ is the set of spokes that i views indirectly in g, i.e.

$$K_i(g) = \{(k, r) : \ r \neq i, r \in N_i(g) \text{ and } (k, r) \text{ is a spoke}\}$$

Property **BS** is not satisfied, because by removing a spoke (k, r) in a network g, the cardinality of $K_i(g)$ may decrease such that $\pi_i(g_{-ij}) \geq 0$ and $\pi_i((g - (k, r))_{-ij}) < 0$.

Clearly, this payoff function satisfies **NA** and **DA**.

For the properties **BT** and **BF** and **BG** notice that $\pi_i(g_{-ij}) = \left|K_i(g)\right| - 1$ for any network g and any agent $j \neq i$. Thus, the payoff $\pi_i(g_{-ij})$ only depends on the number of spokes viewed in g_{-ij}.

Properties **BT** and **BF** are satisfied, because of the following. Let k be an agent in a network g and let j be a topagent in g^k_{-i}. Since i views as least as many spokes in g_{-ij} as in g_{-ik}, **BT** and **BF** are satisfied.

Property, **BG** is satisfied because for any network g and any agent j, the number $\left|K_i(g_{-ij})\right|$ cannot decrease by adding a link to g. □

We already observed that **DE** implies **BT** and **BF**, and not vice versa. In the following theorem, we show that the properties that are needed for Corollary 9, which are **NA**, **DA**, **DE**, **BS** and **BG**, are independent of each other as well.

Theorem 11 *The properties **DA**, **NA**, **DE**, **BG**, **BS** are independent of each other.*

Proof. By Theorem 10 we know that **DA**, **NA**, **BG** and **BS** are independent of each other. Therefore it remains to show that **DA**, **NA**, **BG** and **BS** are independent of **DE**.

Payoff function (3) in Example 6 does not satisfy **DE**, whereas it satisfies all other properties. In the example it was shown that **DA** and **NA** are satisfied.

The other properties, **BG** and **BS**, are trivially satisfied since $\pi_i(g_{-ij}) = 0$ for all g and j. Hence **NA**, **DA**, **BG** and **BS** do not imply **DE**.

To show that **DE** does not imply **DA**, **NA**, **BG** nor **BS**, consider the payoff functions (7), (8), (9) and (10). They do not satisfy **DA**, **NA**, **BG** and **BS** respectively. However, it can be easily checked that these functions do satisfy **DE**.

We conclude that the properties **DA**, **NA**, **DE**, **BG**, **BS** are independent of each other. □

4.4 Relationship with B&G payoff functions

In this subsection, we analyse B&G functions in view of the framework of payoff properties as discussed previously. We prove that B&G functions with owner-homogeneous link costs and heterogeneous profits are orderly and also satisfy **DE**. Then, we prove that B&G functions with heterogenous link costs that satisfy a system of triangle inequalities, are orderly without necessarily satisfying **DE**. Further, we provide several examples of payoff functions that satisfy all properties, while they fall outside the scope of B&G functions.

By defining B&G functions, we assumed that $v_{ii} = 0$. This assumption can be made without loss of generality, since $\pi_i(g_{-i}) = v_{ii}$ for all g, and therefore the transformation $\pi_i'(g) = \pi_i(g) - \pi_i(g_{-i}) = \pi_i(g) - v_{ii}$ can be applied. Hence our assumption $\pi_i(g_{-i}) = 0$ for all g, that we made for general payoff functions, is in line with B&G functions.

In the next lemma, we prove that all B&G functions with heterogeneous link costs satisfy four properties.

Lemma 12 *Let π be a B&G function with heterogeneous link costs. Then π satisfies **DA**, **NA**, **BG** and **BS**.*

Proof.
(**DA**) For each two i-disjoint networks g and g' it holds that $N_i(g) \cap N_i(g') = \{i\}$ and $N_i^d(g) \cap N_i^d(g') = \emptyset$. Since $v_{ii} = 0$, it follows that $\pi_i(g+g') = \pi_i(g) + \pi_i(g')$. Therefore π satisfies **DA**.

(**NA**) If a directed path exists from k to i in network g where link (k,i) does not exist, then $N_i(g) = N_i(g + (k,i))$, and $N_i^d(g) \subset N_i^d(g + (k,i))$. Hence property **NA** is satisfied.

(**BG**) Let g be a network where (j,i) is beneficial. Since $N_i(g_{-ij}) \subseteq N_i((g + (k,r))_{-ij})$ and $N_i^d(g_{-ij}) = N_i^d((g + (k,r))_{-ij}) = 1$, property **BG** is satisfied.

(**BS**) Let g be a network that contains a spoke (k,r). Let (j,i) be beneficial in g. Since $N_i(g_{-ij}) = N_i((g - (k,r))_{-ij})$ and $N_i^d(g_{-ij}) = N_i^d((g - (k,r))_{-ij}) = 1$, link (j,i) is also beneficial in $g - (k,r)$. Hence **BS** is satisfied. □

B&G functions with owner-homogeneous link costs satisfy all properties, as we will see in the following result.

Theorem 13 *Let π be a B&G function with owner-homogeneous link costs, i.e. $c_{ij} = c_i$ for all $i, j \in N$. Then π satisfies* ***DA****,* ***NA****,* ***DE****,* ***BG*** *and* ***BS****.*

Proof. By Lemma 12 it follows that π satisfies **DA**, **NA**, **BG** and **BS**.

Let g be a network where $(j, i) \notin g$, $(k, i) \notin g$, and where a directed path exists from k to j in g_{-i}. Then $N_i(g+(j, i)) \supseteq N_i(g+(k, i))$ and $|N_i^d(g+(j, i))| = |N_i^d(g + (k, i))|$. Hence, property **DE** is satisfied. □

Hence, by Corollary 9 we know that global-Nash networks exist for B&G functions with owner-homogeneous link costs. This is also proved by Billand et al. (2007) and independently by Derks et al. (2007). Global-Nash networks do not exist for B&G functions with heterogeneous link costs in general. Even if these B&G functions are restricted by specific conditions, the existence of global-Nash networks is not guarenteed. This is illustrated by an example provided by Derks et al. (2007). In this example, global-Nash networks do not exist, while the link costs are arbitrarily close to the situation of owner-homogeneity, i.e. $|c_{ij} - c_{ik}| \leq \epsilon$, for all $i, j, k \in N$ and an arbitrarily $\epsilon > 0$.

The existence of local-Nash networks is proved in Theorem 7 for orderly payoff functions. Notice that these payoff functions satisfy **BT** and **BF** instead of **DE** (which implies both of them). In the next theorem we provide conditions for B&G function with heterogeneous link costs such that these functions are orderly.

Theorem 14 *Let π be a B&G function with heterogeneous link costs and profits. If*

$$c_{ij} \leq v_{ij} + min(v_{ik}, c_{ik}), \text{ for all } i, j, k \in N, \tag{11}$$

then π is orderly.

Proof. By Lemma 12, the properties **DA**, **NA**, **BG** and **BS** are satisfied. It remains to prove that π satisfies **BT** and **BF**:
(**BT**) Let link (k, i) be beneficial in g. Then $\sum_{r \in N_i(g_{-ik})} v_{ir} \geq c_{ik}$. If a topagent j exists in the component g^k_{-i}, then either $k = j$ or a directed path from k to j exists. In the first case **BT** is trivially satisfied. In the second case, it follows that $N_i(g_{-ij}) \supseteq N_i(g_{-ik}) \cup \{j\}$. Since $c_{ij} \leq v_{ij} + c_{ik}$ we have

$$\begin{aligned} \pi_i(g_{-ik}) &= \Big(\sum_{r \in N_i(g_{-ik})} v_{ir} \Big) - c_{ik} \\ &\leq \Big(\sum_{r \in N_i(g_{-ik})} v_{ir} \Big) - (c_{ij} - v_{ij}) \\ &\leq \Big(\sum_{r \in N_i(g_{-ij})} v_{ir} \Big) - c_{ij} \\ &= \pi_i(g_{-ij}). \end{aligned}$$

Hence **BT** is satisfied by π.

(**BF**) Let g^k_{-i} be a proper component of g where i has an outgoing link and let link (k,i) be beneficial in g. Let j be a topagent in this component who is farthest away from i. If $k = j$ then **BF** is trivially satisfied. Otherwise a path from k to j exists. Therefore both agents j and k are contained in $N_i(g_{-ij})$. Since $c_{ij} \leq v_{ij} + v_{ik}$ it follows that $\pi_i(g_{-ij}) \geq v_{ij} + v_{ik} - c_{ij} \geq 0$. Hence **BF** is satisfied. $\square$

Hence, by Theorem 7 we know that local-Nash networks exist for B&G functions with heterogeneous link costs that satisfy the triangle conditions of (11). For a full characterization of B&G functions that satisfy the properties of our framework, we refer to Derks and Tennekes (2008a).

Our framework of properties is also satisfied by non B&G payoff functions. Consider the following examples:

$$\begin{aligned}
\pi_i(g) &= |N_i^d(g) \cap T(g_{-i})|; \\
\pi_i(g) &= |K_i(g)|; \\
\pi_i(g) &= |C(g) \cap N_i(g)| - |N_i^d(g)|,
\end{aligned}$$

where $C(g)$ is set of agents that are contained in a directed cycle in g. Recall that $T(g)$ and $K_i(g)$ are respectively defined as the set of topagents in g who do not have outgoing links, and the set of spokes in g that i observes indirectly. These payoff functions are orderly and also satisfy **DE**. Payoff function (3) in Example 6, which is also studied in the proof of Theorem 11, is a non B&G payoff function that is orderly whereas it does not satisfy **DE**.

These payoff functions extend the class of B&G functions in the following way. They do not only consider which agents are (directly) observed, i.e. which agents are contained in the sets $N_i(g)$ and $N_i^d(g)$. They also take other aspects of the network architecture into account. In the given examples, the sets $T(g_{-i})$, $K_i(g)$, and $C(g)$ illustrate this. Also, the set $L(g)$ of 'leaf' agents, i.e. agents without incoming links, that is used in payoff function (3), is a typical example of another aspect of the network architecture.

5 Dynamics

In this section we analyse a dynamic process of iterated local actions that takes place without central coordination. We consider a procedure in which the agents alternately play good local responses. Our choice for good local responses is argued by the fact that agents rather prefer ad-hoc to deliberate decision making in real-life situations, especially in large networks where even finding best local responses takes a lot of effort. Recall that an agent plays a good response, if his payoff does not decrease. If this payoff remains the same, then we say that this agent plays a *neutral response*.

The dynamic procedure that we study in this paper, starts with an arbitrary initial network. Then, one agent is selected at random. One of his good local responses is selected at random, and being played. These steps are repeated. Formally, we define the procedure on base of the following assumptions.

A-1 Let the initial network be a network that is arbitrarily chosen from $\mathcal{G}$.

A-2 At the beginnig of each stage, an agent is selected at random, where each agent has a positive stage independent probability to be chosen.

A-3 At each stage, the agent who is chosen plays a good local response that satisfies the following three assumptions.

A-3a A neutral addition is not allowed.

A-3b A neutral deletion of link (j, i) in network g is only allowed whenever $N_i(g - (j, i)) = N_i(g)$.

A-3c A neutral replacement of (k, i) by (j, i) in network g is only allowed when a directed path exists from k to j in network g_{-i}.

He choses a good local response at random, where all allowed good local responses have a positive probability to be chosen that only depends on the network.

In this section we prove that this procedure converges to a local-Nash network. We say that the procedure *converges* to a network g if this network is reached, and furthermore, if a pass is the only allowed good local response for each agent i with respect to g.

We need assumptions A-3a to A-3c in order to prevent the following situation. Consider a game where an agent i is present, such that $\pi_i(g) = 0$ for all networks g. We refer to this agent as a *zero-agent.* Consider a local-Nash network that has been reached by the dynamic procedure. A zero-agent may perform randomly chosen neutral local responses, such that the obtained network is not local-Nash again. Notice here that for any game where zero-agents exist, strict local-Nash networks do not exist.

Assumptions A-3a to A-3c control neutral responses in a local way, i.e. without central coordination. Besides, these assumptions are intuitive in the context of one-way flow models. An addition is only allowed when it is a strictly improvement. A deletion is only allowed when the set of observed agents does not change, for instance when a spoke is deleted. Finally, a neutral replacement is only allowed when it is downstream, i.e. when property **DE** is applicable.

We prove that the procedure converges to a local-Nash network with probability 1. In order to prove to this, we show that a finite sequence of good local responses exists which can be applied to an arbitrary network that leads to a local-Nash network. This sequence starts with actions such that the initial network is reshaped to a proper and beneficial network. From there, we re-use the result of the proof of Theorem 7 which states that if a proper and beneficial network is not local-Nash, then another proper and beneficial network exists with a higher connection number. Iteratively using this result, we obtain a network with a maximal connection number, which implies that this network is local-Nash.

In the following lemma we show that a sequence of good local responses exists that leads to a proper and beneficial network.

Lemma 15 *Let π satisfy **DA**, **NA** and **DE**. Then, for any network in $\mathcal{G}$, there exists a finite sequence of good local responses that leads to a proper and beneficial network.*

Proof.

Step 1 Let $g \in \mathcal{G}$. First we make g proper by applying good local responses. If g is already proper, then continue to step 2. Otherwise an agent i exists in g who has at least two outgoing links, say (i,j) and (i,k) (see Figure 8).

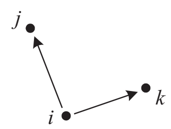

Figure 8: An agent with two outgoing links

Two cases are distinguished:

A: There is a directed path from i to an agent ℓ with outdegree 0, starting with link (i,k). The property **DE** implies that the link (i,j) may be replaced by (ℓ,j). This action decreases the total outdegree of the agents with multiple outgoing links.

B: None of the directed paths starting with link (i,k) end at an agent with outdegree 0. Either there is a cycle C containing (i,k), or there is a directed path starting with link (i,k) and ending at an agent ℓ on a (directed) cycle. In the latter case we may apply the property **DE** and replace link (i,j) by (ℓ,j). It is therefore no loss of generalization to assume a cycle with (i,k) in it.

We distinguish four subcases:

1: Agent j is on cycle C. Then a directed path exists from i to j and hence the link (i,j) can be deleted by **NA**.

2: There is a directed path from i to an agent with outdegree 0, and starting with link (i,j). Case **A** addresses this situation.

3: There is a cycle C' containing (i,j). Going in the opposite direction over C', let ℓ be the last agent on this cycle who is also on the cycle C through (i,k) (see Figure 9). Using property **DE** we may replace link (i,k) by the link (ℓ,k), so that we can assume that both cycles C and C', have only agent i in common. This situation is depicted in Figure 10.

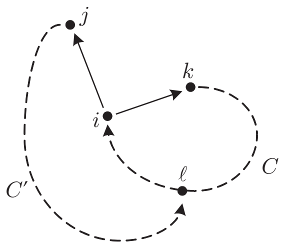

Figure 9: Situation of case 3

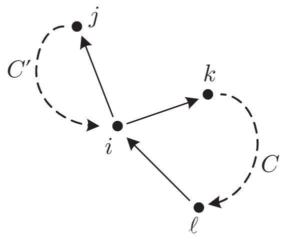

Figure 10: Situation of case 3 continued

Let agent ℓ be such that (ℓ, i) is on cycle C (ℓ may be the agent k). Now, replace (i, j) with (ℓ, j). This is a good local response by **DE** since there is a directed path from i to ℓ, without visiting j. After this replacement, the link (ℓ, i) can be deleted by **NA** since there is a directed cycle in which i, k, ℓ, j are visited in this order, and hence (ℓ, i) is a spoke.

4: there is a directed path starting with link (i, j) and ending at an agent ℓ on a (directed) cycle. Then agent ℓ is not on the cycle through (i, k). (Otherwise, we would have obtained a cycle containing (i, j), and this is already taken care of in case 3.) Using property **DE**, we may replace link (i, k) by the link (ℓ, j), and by this action we arrive at the situation treated in case 3.

As long as there are agents with outdegree greater than 1, g is not proper, and hence this step can be repeated. Each time it is repeated, the outdegree of one agent is reduced without changing the outdegrees of the other agents. Therefore, after a finite number of repetitions we obtain a proper network.

Step 2 Let g' be the proper network that results from step 1. If g' is not beneficial, then a non-beneficial link (j, i) exists in g'. Since g' is proper, by **DA** the deletion of (j, i) is a good local response. Obviously, g' remains proper after this deletion. Such deletions can be applied repeatedly until we obtain a proper and beneficial network g''. $\square$

The next lemma shows that there exists a finite sequence of good local responses that starts with a proper and beneficial network and that leads to a local-Nash network. From the proof of Theorem 7 we may deduce that from a non local-Nash network which is proper and beneficial, a network can be constructed which is also proper and beneficial but which has a higher κ-value. We show that this construction can be done by a sequence of good local responses.

Lemma 16 *Let π be an orderly payoff function that satisfies **DE**. Let g be a proper and benficial network. There exists a finite sequence of good local responses that leads to a local-Nash network.*

Proof. Suppose that g is not local-Nash. Since g is proper and beneficial, we know by the proof of Theorem 7 that a network can be obtained with a higher connection number. We show that we can obtain this network by applying good local reponses.

Consider the networks g', g'' and g''' as defined in (4), (5), and (6). Network g' is obtained from g by a strictly improving addition, which is trivially a good local response. Network g'' is obtained from g' by a replacement of (j, k) by (r, k) where a directed path from j to r exists in g. By **DE**, this is also a good local response. Finally, network g''' is obtained from g'' by a deletion of spoke (r, s) which is a good local response by **NA**. Observe that g''' is proper and benefical. Therefore, if g''' is not local-Nash, we can repeat these good local responses until we obtain a local-Nash network. At each iteration, the connection number increases. Since this number is bounded by $n(n-1)$, we obtain a local-Nash network in a finite number of iterations. $\square$

Combining Lemma's 15 and 16 we obtain a sequence of networks that starts with an arbitrary initial network and ends with a local-Nash network. In the next theorem we show that our procedure always reaches a local-Nash network.

Theorem 17 *Let π be an orderly payoff function that satisfies **DE**, and let the dynamic procedure be as defined by assumptions A-1 to A-3c. Then this procedure converges to a local-Nash network with probability 1.*

Proof. First we prove that the procedure reaches a local-Nash network with probability 1, and then we prove that it also converges to this network with probability 1. By Lemma's 15 and 16 we know that from an arbitrary network in $\mathcal{G}$ a finite sequence of good local responses exists, such that the obtained network is proper and local-Nash. It is easily verified that these good local responses satisfy assumptions A-3a to A-3c:

- the only additions in this sequence are strictly improving ones;
- each deletion is either validated as a good local response by **NA** (and hence it satisfies assumption A-3b), or it is a deletion of a non-beneficial link in a proper network which is a strictly improving deletion by **DA**;
- all replacements in this sequence are validated as good local responses by **DE** and hence they satisfy assumption A-3c.

Hence, any sequence that is constructed in the proofs of Lemma's 15 and 16 satisfies the assumptions A-1 to A-3c.

By the contruction of such a sequence, we know that each network in $\mathcal{G}$ appears at most once in this sequence. Therefore, we conclude that the length of this sequence is upperbounded by M, which is defined as the number of networks in $\mathcal{G}$.

At any stage, each agent has a strictly positive probability to be chosen (assumption A-2), and each allowed good local response has a strictly positive probability to be chosen (assumption A-3). Therefore, the probability that such a sequence will be played is lowerbounded by a stricty positive probability ϵ.

The probability that the dynamic procedure does not reach a local-Nash network after M steps is lower than $1 - \epsilon$. If it does not reach a local-Nash network after M steps, then from the last network, another such a sequence exists that leads to a local-Nash network. Hence, the probability that the dynamic procedure does not reach a local-Nash network after $2M$ steps is lower than $(1 - \epsilon)^2$, and after kM steps lower than $(1 - \epsilon)^k$, with k being a strictly positive natural number. Hence we conclude that this probability converges to 0 as k becomes larger. Therefore, this procedure reaches a local-Nash network with probability 1.

Let g be the local-Nash network obtained by the dynamic procedure. By the proof of Lemma 16 we know that g is proper. Since g is local-Nash, it can only be modified by neutral responses. Let i be an agent who can apply a neutral response to g. We know by assumption A-3a that this action cannot be a neutral addition.

Suppose that this action is a deletion. Since g is proper, each deletion strictly reduces the set of observed agents. By assumption A-3b, these deletions are not allowed. Hence we conclude that this action cannot be a deletion.

Suppose that this action is a replacement. By assumption A-3c, a neutral replacement of (k, i) by (j, i) is only allowed when a directed path exists from k to j that does not visit agent i. In that case, agent k has two outgoing links: (k, i) and a link on the path from k to j. This contradicts that g is proper.

Hence we conclude that the only neutral response that can be applied to g is a pass. Therefore, g is the final network in the dynamic procedure. $\square$

Notice that each local-Nash network that is the final network of procedure defined by assumptions A-1 to A-3c is also global-Nash, because it is proper and therefore Theorem 5 applies here. Hence we have the following corollary.

Corollary 18 *Let π be an orderly payoff function that satisfies **DE**, and let the dynamic procedure be as defined by assumptions A-1 to A-3c. Then this procedure converges to a global-Nash network with probability 1.*

6 Conclusion

In this paper, we have studied a dynamic model of unilateral network formation. We have extended the literature on non-cooperative network formation in two ways. First, we introduced a local approach, where agents are restricted to play local actions. Second, we developed a framework of axiomatic payoff properties.

We proved the existence of local-Nash and global-Nash networks for games with payoff functions that satisfy these properties. Further, we prove that our iterative procedure of local actions always converges to a local-Nash network, which is also global-Nash. Thus, we know that it converges in a finite number of iterations. However, we would like to find upperbounds for this number. Experiments on our dynamic procedure may also provide insights about the speed of convergence.

Our framework of properties is inspired by the one-way flow model that is introduced by Bala and Goyal (2000a). Besides the one-way flow model, Bala and Goyal (2000a) introduced another model, called the two-way flow model. The only difference between the one-way and the two-way flow model is that in the latter, profits flow in both directions of the links. The two-way flow model is also studied by Galeotti et al. (2006) and Haller et al. (2007). Unfortunately, our results do not apply to the two-way flow model, since one of our properies, **BG**, is not satisfied here. To show this, consider the network $g = \{(j,i)\}$ where (j,i) is beneficial. Now consider the network $g' = g + (i,j)$. Here, agent i also observes j via (i,j), which implies that his own link (j,i) is not beneficial in g'. For further reseach, it would be interesting to develop a framework of payoff properties that is inspired by the two-way flow model.

Several enhancements of models of networks formation have been studied in literature. For instance the role of decay, where the profits that agent i receives from being connected to j depends on the path length, or the role of link reliability where each link of the formed network is functioning with a certain probability. It would be interesting to develop axiomatic payoff properties for these models as well.

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