JOURNAL OF OPTIMIZATION THEORY AND APPLICATIONS: Vol. 105, No. 3, pp. 567–588, JUNE 2000

Polytope Games[1]

R. BHATTACHARJEE,[2] F. THUIJSMAN,[3] AND O. J. VRIEZE[4]

Abstract. Starting from the definition of a bimatrix game, we restrict the pair of strategy sets jointly, not independently. Thus, we have a set $P \subset S_m \times S_n$, which is the set of all feasible strategy pairs. We pose the question of whether a Nash equilibrium exists, in that no player can obtain a higher payoff by deviating. We answer this question affirmatively for a very general case, imposing a minimum of conditions on the restricted sets and the payoff. Next, we concentrate on a special class of restricted games, the polytope bimatrix game, where the restrictions are linear and the payoff functions are bilinear. Further, we show how the polytope bimatrix game is a generalization of the bimatrix game. We give an algorithm for solving such a polytope bimatrix game; finally, we discuss refinements to the equilibrium point concept where we generalize results from the theory of bimatrix games.

Key Words. Game theory, bimatrix games, Nash equilibria, restricted games.

1. Introduction

We consider in this paper classical noncooperative two-person games with one interesting distinction: the players strategies are restricted.

In noncooperative, two-person games, the pure actions of the players are enumerated $\{1, \ldots, m\}$ and $\{1, \ldots, n\}$ and then their mixed actions are defined as

$$S_m := \left\{ x \in \mathbb{R}^m \middle| x \geq 0 \text{ and } \sum_{i=1}^{m} x_i = 1 \right\},$$

$$S_n := \left\{ y \in \mathbb{R}^n \middle| y \geq 0, \sum_{j=1}^{n} y_j = 1 \right\}.$$

[1]This paper is dedicated to David Luenberger. We are grateful for the valuable remarks of an anonymous referee on an earlier version of the paper.

[2]Research Assistant, Department of Mathematics, Boston University, Boston, Massachusetts.

[3]Professor, Department of Mathematics, Maastricht University, Maastricht, Netherlands.

[4]Professor, Department of Mathematics, Maastricht University, Maastricht, Netherlands.

0022-3239/00/0600-0567$18.00/0

Formally, such a game is characterized by the tuple $\langle R_1, R_2, S_m, S_n\rangle$, where R_1 and R_2 are the payoff functions for the two players.

It can be shown that a restriction of the strategy sets such that the set of all feasible strategies (x, y) is of the form $X\times Y$, where $X\subset S_m$, $Y\subset S_n$, and both are polytopes, is equivalent to some bimatrix game where the pure actions in that game are the extreme points of X and Y. This can be done because the players still have independent strategy sets.

We consider another class of restrictions; we restrict the joint strategy set $S_m\times S_n$ to a set $C\subset S_m\times S_n$. Thus, the players strategies are not independent of each other. The question arises of whether two players, choosing a pair of points $(x, y)\in C\subset S_m\times S_n$, where C is convex and compact, can reach a Nash equilibrium in the sense that both players cannot achieve a better payoff within C, if the other player stays with his strategy.

Formally, this game can be characterized as follows. Let $C\subset S_m\times S_n$ be convex and compact. Further, define payoff functions $R_1(x, y): C\to\mathbb{R}$ and $R_2(x, y): C\to\mathbb{R}$ to be concave and continuous. The two players play a game in normal form with compact action spaces S_m and S_n. The two players choose strategies $x\in S_m$ and $y\in S_n$, without knowledge of the other players choice. If $(x, y)\notin C$, then the payoff is $(-\infty, -\infty)$; if $(x, y)\in C$, then the payoff is $(R_1(x, y), R_2(x, y))$.

This is a noncooperative two-person general-sum game; therefore, the concept of a Nash equilibrium, proposed by Nash in Ref. 1, is still valid. If there exists $x\in S_m$ such that, $(x, y)\notin C$, for all $y\in S_n$, then there might exist a Nash equilibria with payoff $(-\infty, -\infty)$. It is important to see that it is not possible for one player to receive negative infinite payoff, while the other receives a finite payoff. The question that now remains is whether there are Nash equilibria with finite payoff, assuming that the players prefer a finite payoff. This is equivalent to finding Nash equilibria over C, i.e., allowing only strategies $(x, y)\in C$ to be played. We call this a restricted game and show the existence of Nash equilibria over C. We look at a special class of restricted games, polytope bimatrix games, in which the restrictions on the strategy set are linear and the payoff functions are bilinear. These polytope bimatrix games are in fact a generalization of bimatrix games; we discuss the similarities and differences.

An example of a restricted game can be found in a situation when there is a finite resource. Consider two neighboring countries. The clean air which they share is a finite resource. There exists an upper bound on the sum of allowable pollution; e.g., if both countries together pollute over this limit, they both suffer. If a payoff function is defined with respect to the health of the citizens and if the level of pollution is a strategy, then for some tuple of strategies, the payoff is a large negative number for both parties. This is undesirable; hence, the two countries search for equilibrium strategies that do not have this large negative payoff.

The organization of the paper is as follows. In Section 2, we give the basic definitions for playing the game with restricted strategies and the definition of equilibrium strategies. These are as general as possible. In Section 3, we show the main result, the existence of equilibrium strategies. This is done with the help of the fixed-point theorem of Kakutani. Again, we impose minimal conditions on the restricted set and the payoffs so as to arrive at a general statement about the object of our study, the polytope bimatrix game. In Section 4, we derive how the best reply sets of a bimatrix game are projected on a polytope bimatrix game with the same payoff matrices, but with restrictions on the original set of strategies. Section 5 shows how to solve a polytope bimatrix game by means of a linear complementarity problem. In Section 6, we give some results on the structure of the set of equilibria using results from Section 5. Section 7 deals with refinement to the equilibrium concept. Here, we have extended existing refinements from the theory of bimatrix games to polytope games.

2. Definitions and Preliminary Results

Consider a convex and compact set $C \subset S_m \times S_n$ and payoff mappings $R_1(x,y)\colon C \to \mathbb{R}$, $R_2(x,y)\colon C \to \mathbb{R}$. Players I and II play a game in normal form with actions $x \in S_m$ and $y \in S_n$. If $(x,y) \notin C$, their payoffs are $(-\infty, -\infty)$; if $(x,y) \in C$, their payoff are $(R_1(x,y), R_2(x,y))$. One can assume that both players desire a finite payoff. This is equivalent to the following definition.

Definition 2.1. Let the set $C \subset S_m \times S_n$ be nonempty, convex, and compact. Further, let the payoff functions $R_1\colon S_m \times S_n \to \mathbb{R}$ and $R_2\colon S_m \times S_n \to \mathbb{R}$ be continuous and concave on C. Players I and II can play only strategies x and y respectively, such that $(x,y) \in C$. Then, their payoffs are $R_1(x,y)$ and $R_2(x,y)$ respectively. We say that players I and II play a restricted game, denoted by $\langle C, R_1, R_2 \rangle$.

Since both players want a finite payoff they wish to settle for an equilibrium which has a finite payoff. Hence, the equilibrium can be formalized via the following definition.

Definition 2.2. A strategy pair $(x,y) \in C$ is called a Nash equilibrium for the restricted game $\langle C, R_1, R_2 \rangle$ if

$$R_1(x,y) \geq R_1(\hat{x},y), \qquad \text{for all } (\hat{x},y) \in C,$$

$$R_2(x,y) \geq R_2(x,\hat{y}), \qquad \text{for all } (x,\hat{y}) \in C.$$

Given that, if $(x, y) \notin C$, the payoff is $(-\infty, -\infty)$, a Nash equilibrium for $\langle C, R_1, R_2 \rangle$ is also a Nash equilibrium in the classical sense over $S_m \times S_n$, since

$$R_1(x, y) > R_1(\hat{x}, y), \qquad \text{for all } (\hat{x}, y) \notin C.$$

In later sections, the object of our investigations is a special class of restricted games, the polytope bimatrix game.

Definition 2.3. A restricted game $\langle C, R_1, R_2 \rangle$ is known as a polytope bimatrix game (PBG), if C is a polytope in $\mathbb{R}^{m \times n}$ and the payoffs are $R_1 = xAy^t$, $R_2 = xBy^t$, for some $A, B \in \mathbb{R}^{m \times n}$. The polytope bimatrix game is denoted by $\langle P, A, B \rangle$ and its set of equilibria by $E_P(A, B)$.

3. Equilibrium Points

In this section, we show that an equilibrium in the sense of Definition 2.2 always exists for a restricted game. From here on, we disregard strategy pairs which are not in the restricted set.

Theorem 3.1. Let $\langle C, R_1, R_2 \rangle$ be a restricted game. Then, there exists $(x, y) \in C$ such that (x, y) is a Nash equilibrium.

The classical proof for existence of Nash equilibria in finite noncooperative games fails in this case, because the Cartesian product of the best replies for a strategy pair need not be in the restricted set. We include the following proof only to show how to circumvent this problem.

Once we have overcome this obstacle, the proof is nearly identical to the existence proof for classical bimatrix games.

We employ some results about semicontinuous functions and also need the following definition.

Definition 3.1. Let $\langle C, R_1, R_2 \rangle$ be a restricted game. For $y \in S_n$, define

$$X(y) := \{x \in S_m \mid (x, y) \in C\};$$

for $S \subset S_n$, let

$$X(S) := \bigcup_{y \in S} X(y).$$

For $x \in S_m$, define

$$Y(x) := \{y \in S_n \mid (x, y) \in C\};$$

for $S \subset S_m$, let

$$Y(S) := \bigcup_{x \in S} Y(x).$$

Proof of Theorem 3.1. Let $(\tilde{x}, \tilde{y}) \in C$. Define

$$BR_2(\tilde{x}) := \{y \in Y(\tilde{x}) \,|\, R_2(\tilde{x}, y) \geq R_2(\tilde{x}, \bar{y}), \text{ for all } \bar{y} \in Y(\tilde{x})\},$$

$$BR_1(\tilde{y}) := \{x \in X(\tilde{y}) \,|\, R_1(x, \tilde{y}) \geq R_1(\bar{x}, \tilde{y}), \text{ for all } \bar{x} \in X(\tilde{y})\}.$$

So, $BR_2(\tilde{x})$ is the set of best replies of player 2 against $\tilde{x}$ in C. It follows that

$$BR_2(x)[BR_1(y)] \text{ is upper semicontinuous in } x\ [y].$$

Now, we define

$$\phi: C \times C \to C \times C,$$

$$\phi(x_1, y_1, x_2, y_2) \in (BR_1(y_2) \times \{y_2\} \times \{x_1\} \times BR_2(x_1)).$$

Standard results yield that ϕ is compact-valued, convex-valued, and upper semicontinuous, from which it follows that, with the fixed-point theorem of Kakutani (Ref. 2), there exists $(x_1^*, y_1^*, x_2^*, y_2^*)$ such that

$$(x_1^*, y_1^*, x_2^*, y_2^*) \in \phi(x_1^*, y_1^*, x_2^*, y_2^*).$$

This implies that

$$x_1^* = x_2^* \in BR_1(y_2^*) = BR_1(y_1^*),$$

$$y_1^* = y_2^* \in BR_2(x_2^*) = BR_2(x_1^*).$$

Hence, it follows that $(x_1^*, y_1^*) = (x_2^*, y_2^*)$ is an equilibrium. □

In the following example, we show how a restricted game looks and which points form equilibria. We have taken a simple shape (the circle) and have restricted our strategy pairs to points enclosed by this circle. The payoff is defined in two matrices.

Example 3.1. Consider the bimatrix game defined by

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

The strategy space $S_m \times S_n$ is restricted to

$$C := \{(x, y) \in S_m \times S_n \,|\, (x_2 - 1/2)^2 + (y_2 - 1/2)^2 \leq 1/4\}.$$

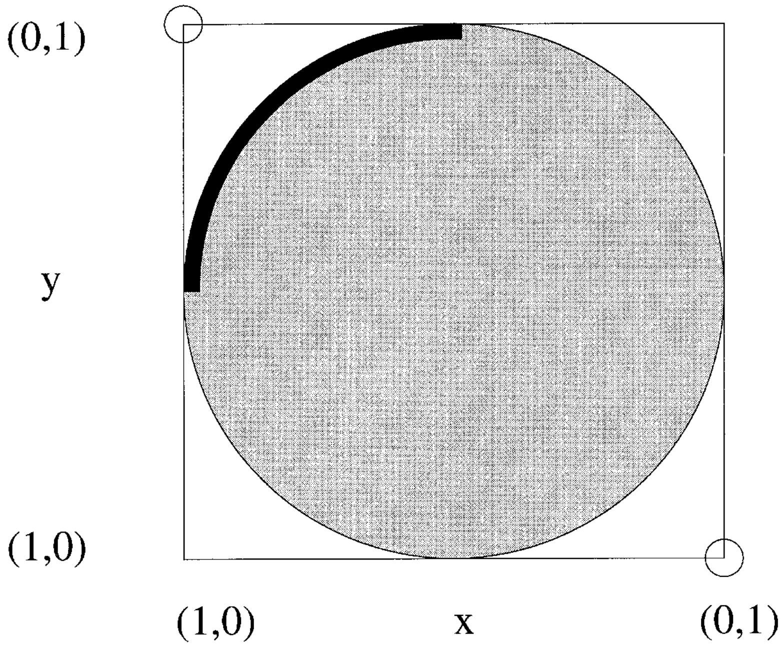

Fig. 1. Example 3.1.

This is a restricted game with

$$R_1 := xAy^t, \qquad R_2 := xBy^t$$

continuous and concave, and C convex and compact. The set of all equilibria is

$$E(A, B) = \{(x, y) \,|\, (x^2 - 1/2)^2 + (y_2 - 1/2)^2 = 1/4\}$$
$$\cap \{(x, y) \,|\, x_2 \leq 1/2, y_2 \geq 1/2\}.$$

Figure 1 shows the original set $S_m \times S_n$. The restricted strategy set C is shown shaded and the equilibria are dark.

4. Projecting the Best Replies

Having shown the existence of equilibria for a very general case of games, we now restrict our study to polytope bimatrix games. In fact, the polytope bimatrix game is a generalization of the bimatrix game. This can be seen by choosing

$$P = S_m \times S_n.$$

Then,

$$(A, B) \equiv \langle P, A, B \rangle.$$

When investigating a polytope bimatrix game, the natural place to start is the original bimatrix game, obtained by removing the restrictions on the strategy set. Indeed, the common factor between a polytope bimatrix game and its original bimatrix game are the payoff functions, and these will determine also the way that the set of equilibria of the bimatrix game (A, B), denoted by $E(A, B)$, influences the structure of the set $E_P(A, B)$.

We use the fact that both $E(A, B)$ and $E_P(A, B)$ consist of strategy pairs that are simultaneous solutions to a pair of linear programming problems. In both cases, the pairs of LPs have identical objective functions, but the LPs for the two cases have different feasible sets.

Starting from solutions to the pair of LPs in the case of a bimatrix game, we use the gradients of the payoff functions to project the solutions onto the new feasible set for the polytope bimatrix game. In this way, we obtain a pair of strategies that are simultaneous solutions to the pair of linear programming problems for the polytope bimatrix game.

Definition 4.1. Let (A, B) be a bimatrix game. For $(x, y) \in S_m \times S_n$, define

$$B_I(y) := \{x \in S_m \mid xAy^t \geq \bar{x}Ay^t, \text{ for all } \bar{x} \in S_m\},$$
$$B_{II}(x) := \{y \in S_n \mid xBy^t \geq xB\bar{y}^t, \text{ for all } \bar{y} \in S_n\}.$$

These are the best reply sets for a bimatrix game. The following is a well-known result.

Corollary 4.1. Let (A, B) be a bimatrix game. Then,

$$E(A, B) = \bigcup_{x \in S_m} (\{x\} \times B_{II}(x)) \cap \bigcup_{y \in S_n} (B_I(y) \times \{y\}).$$

Now, consider the polytope bimatrix game $\langle P, A, B \rangle$, with the same payoff functions xAy^t and xBy^t, and consider how the best reply sets are projected onto the polytope.

Definition 4.2. Let $\langle P, A, B\rangle$ be a polytope bimatrix game. For $(x, y) \in P$, define

$$OH_I(y) := \left\{x \in \mathbb{R}^m \,\middle|\, xAy^t = \max_{\bar{x} \in S_m} \bar{x}Ay^t\right\},$$

$$OH_{II}(x) := \left\{y \in \mathbb{R}^n \,\middle|\, xBy^t = \max_{\bar{y} \in S_n} xB\bar{y}^t\right\}.$$

These sets are hyperplanes in $\mathbb{R}^m$ and $\mathbb{R}^n$. They are the isoclines of the objective functions for the linear problem of maximizing the player I payoff, while keeping y fixed and vice versa. These are projected onto the polytope.

It is clear that

$$B_I(y) \subset OH_I(y),$$
$$B_{II}(x) \subset OH_{II}(x).$$

Now, the hyperplanes $OH_I(y) \times \{y\}$ and $\{x\} \times OH_{II}(x)$ are projected onto the polytope P. Define

$$O_I(y) := \{x \in X(y) \,|\, \mathrm{dist}((x, y), (OH_I(y) \times \{y\})) = \min_{\bar{x} \in X(y)} \mathrm{dist}((\bar{x}, y), (OH_I(y) \times \{y\}))\},$$

$$O_{II}(x) := \{y \in Y(x) \,|\, \mathrm{dist}((x, y), (\{x\} \times OH_{II}(x))) = \min_{\bar{y} \in Y(x)} \mathrm{dist}((x, \bar{y}), (\{x\} \times OH_{II}(x)))\}.$$

Theorem 4.1. Let $\langle P, A, B\rangle$ be a PBG; let

$$X := \bigcup_{y \in S_n} X(y) \quad \text{and} \quad Y := \bigcup_{x \in S_m} Y(x).$$

Then,

$$E_P(A, B) = \bigcup_{x \in X} (\{x\} \times O_{II}(x)) \cap \bigcup_{y \in Y} (O_I(y) \times \{y\}).$$

Proof. $(\subseteq)$ Let

$$(x, y) \in E_P(A, B).$$

Then, it follows that

$$x = \arg\max_{\bar{x} \in X(y)} \bar{x}Ay^t,$$

$$y = \arg\max_{\bar{y} \in Y(x)} xB\bar{y}^t.$$

The point in $X(y)$ with minimal distance to $OH_I(y)$ has maximal projection on the normal Ay^t of $OH_I(y)$. Since the length of the projection onto Ay^t is $xAy^t/\|Ay^t\|$, it follows that

$$\mathrm{dist}((x,y),(OH_I(y)\times\{y\})) = \min_{\bar{x}\in X(y)} \mathrm{dist}((\bar{x},y),(OH_I(y)\times\{y\})).$$

($\supseteq$) Let

$$(x,y)\in(\{x\}\times O_{II}(x))\cap(O_I(y)\times\{y\}).$$

Then, $x\in O_I(y)$, and since

$$\max_{\bar{x}\in X(y)} \bar{x}Ay^t \leq \max_{\bar{x}\in S_m} \bar{x}Ay^t,$$

we find that x minimizes the distance to the isocline with the maximum possible payoff. Therefore, x is solution to the following problem: maximize $x(Ay^t)$ subject to $x\in X(y)$. □

5. Solving Polytope Games

Here, we use the knowledge gained in the previous section regarding the gradients of the payoff functions. This helps us characterize equilibrium points by means of a linear complementarity problem. First, we recall the concept of the Kuhn–Tucker conditions.

Theorem 5.1. Consider a linear programming problem with finite solution,

$$\begin{aligned} &\max \quad x\cdot c^t,\\ &\text{s.t.} \quad xA\leq b,\\ &\qquad\quad x\geq 0,\end{aligned}$$

where $A\in\mathbb{R}^{m\times n}$, x, c, $0\in\mathbb{R}^n$, and $b\in\mathbb{R}^m$; a vector $\tilde{x}$ is an optimal solution if and only if the following Kuhn–Tucker conditions are fulfilled: There exist $\mu\in\mathbb{R}^m$ and $u\in\mathbb{R}^n$ such that

$$\begin{aligned} &\mu, u\geq 0,\\ &c=\mu A-u,\\ &u\cdot x^t=0,\\ &\mu\cdot(Ax-b)^t=0.\end{aligned}$$

The vectors μ and u are also called Lagrange multipliers. Thus, if the Kuhn–Tucker condition is fulfilled, the gradient c of the objective function lies in the convex cone of the active restriction, because of the complementarity conditions. This implies that any change to improve the objective function leads outside the feasible set. For a proof, we refer to Luenberger (Ref. 3).

The next theorem uses this idea to characterize equilibrium points.

Lemma 5.1. Let $\langle P, A, B\rangle$ be a PBG. Then, $(x, y)\in E_P(A, B)$ if and only if

$$\text{for all } \varphi\in\mathbb{R}^m \text{ with } (\varphi A y^t) > 0, \text{ we have } (x+\varphi, y)\notin P;$$

$$\text{for all } \psi\in\mathbb{R}^n \text{ with } (xB\psi^t) > 0, \text{ we have } (x, y+\psi)\notin P.$$

Proof. **(⇒)** Let $(x, y)\in E_P(A, B)$. Then,

$$x(Ay^t)\geq\bar{x}(Ay^t), \qquad \text{for all } \bar{x}\in X(y).$$

Assume that there exists $\varphi\in\mathbb{R}^m$ such that $\varphi Ay^t > 0$ and $(x+\varphi, y)\in P$. Then,

$$(x+\varphi)Ay^t = xAy^t + \varphi Ay^t > xAy^t.$$

But this means that there exists

$$\bar{x} = (x+\varphi) \text{ such that } x(Ay^t) < \bar{x}(Ay^t), \text{ with } (\bar{x}, y)\in P,$$

which is a contradiction to $(x, y)\in E_P(A, B)$. So, for all $\varphi\in\mathbb{R}^m$ such that $\varphi Ay^t > 0$, we have $(x+\varphi, y)\notin P$.

(⇐) For all $\varphi\in\mathbb{R}^m$ such that $\varphi Ay^t > 0$ implies $(x+\varphi, y)\notin P$. Assume that there exists $\bar{x}\in X(y)$ such that $\bar{x}(Ay^t) > x(Ay^t)$. Let

$$\varphi\in\mathbb{R}^m \text{ such that } \bar{x} = x+\varphi.$$

Since

$$(x+\varphi)Ay^t = xAy^t + \varphi Ay^t,$$

it follows that $\langle\varphi, (Ay^t)\rangle > 0$. Finally,

$$\langle\varphi, (Ay^t)\rangle > 0 \quad \text{and} \quad (x+\varphi, y) = (\bar{x}, y)\in P,$$

and this is a contradiction to $(x+\varphi, y)\notin P$, so $(x, y)\in E_P(A, B)$. □

This implies that, at an equilibrium, neither player can find a feasible change of strategy to increase his payoff. In fact, the condition in Lemma 5.1 is equivalent to the Kuhn–Tucker conditions, which state that the only improving directions point out of the feasible set.

To use Theorem 5.1 to solve our problem, we will need the fact that, because the set P is a polytope, it can be characterized as a system of linear inequalities.

Lemma 5.2. Let $\langle P, A, B\rangle$ be a PBG. Then, there exist a matrix $M\in \mathbb{R}^{l\times m+n}$ and a vector $b\in\mathbb{R}^l$ such that

$$P=\left\{(x,y)\in\mathbb{R}^{m+n}\,\middle|\, M\begin{bmatrix}x^t\\ y^t\end{bmatrix}\leq b^t \text{ and } x,y\geq 0\right\},$$

where

$$M=\begin{bmatrix}M_e\\ M_p\end{bmatrix},\qquad b^t=\begin{bmatrix}b_e\\ b_p\end{bmatrix},$$

with

$$M_e=\begin{bmatrix}-1 & \dots & -1 & 0 & \dots & 0\\ 1 & \dots & 1 & 0 & \dots & 0\\ 0 & \dots & 0 & -1 & \dots & -1\\ 0 & \dots & 0 & 1 & \dots & 1\end{bmatrix},\qquad b_e=\begin{bmatrix}-1\\ 1\\ -1\\ 1\end{bmatrix},$$

and where M_p and b_p have to be chosen appropriately.

Observe that, for $\tilde{y}\in Y$,

$$X(\tilde{y})=\left\{(x,\tilde{y})\in\mathbb{R}^{m+n}\,\middle|\, M\begin{bmatrix}x^t\\ \tilde{y}^t\end{bmatrix}\leq b^t, x\geq 0\right\}.$$

The matrix M is not unique and we consider the smallest matrix for the sake of simplicity.

Now we can formalize the condition in Theorem 5.1 in the form of the Kuhn–Tucker conditions. This yields the following theorem.

Theorem 5.2. Let $\langle P, A, B\rangle$ be a PBG, where $A, B\in\mathbb{R}^{m\times n}$ and M, b are as in Lemma 5.2. Then, $(x,y)\in E_P(A,B)$ if and only if there exist vectors $\mu,\nu\in\mathbb{R}^l$, $u_1\in\mathbb{R}^m$, $u_2\in\mathbb{R}^n$ with

$$\mu,\nu,u_1,u_2\geq 0,$$

$$(Ay^t)=\sum_{i=1}^{l}\mu_i\begin{bmatrix}M_{i1}\\ \vdots\\ M_{im}\end{bmatrix}-u_1^t,$$

$$x\cdot u_1^t=0,$$

$$\mu\left[M\begin{bmatrix}x^t\\ y^t\end{bmatrix}-b^t\right)=0,$$

$$(xB)^t = \sum_{i=1}^{l} \nu_i \begin{bmatrix} M_{im+1} \\ \vdots \\ M_{im+n} \end{bmatrix} - u_2^t,$$

$$y \cdot u_2^t = 0,$$

$$\nu\left(M\begin{bmatrix} x^t \\ y^t \end{bmatrix} - b^t\right) = 0.$$

Proof. By the definition of an equilibrium point, it follows immediately that $(\tilde{x}, \tilde{y})$ is an equilibrium point if and only if $\tilde{x}$ respectively $\tilde{y}$ solve the following two equations:

$$\max_{x \in X(\tilde{y})} xA\tilde{y}^t = \tilde{x}A\tilde{y}^t,$$

$$\max_{y \in Y(\tilde{x})} \tilde{x}By^t = \tilde{x}B\tilde{y}^t.$$

Using Lemma 5.1, it follows that $\tilde{x}$ respectively $\tilde{y}$ solve these two equations if and only if vectors μ, $\nu \in \mathbb{R}^l$, $u_1 \in \mathbb{R}^m$, $u_2 \in \mathbb{R}^n$ exist that satisfy the equations of the theorem. $\square$

Now, we have a set of equations whose solutions are Nash equilibria. As a direct consequence, we can describe a Nash equilibrium as a linear complementarity problem (LCP).

Corollary 5.1. Let $\langle P, A, B \rangle$ be a PBG. Then, $(x, y) \in E_P(A, B)$ if there exist μ, ν, u_1, u_2, v such that $(x, y, \mu, \nu, u_1, u_2, v)$ is a solution to the LCP

$$\begin{bmatrix} u_1^t \\ u_2^t \\ v^t \\ v^t \end{bmatrix} = \begin{bmatrix} 0 & -A & (M_{1,\dots,l,1,\dots,m})^t & 0 \\ -B^t & 0 & 0 & (M_{1,\dots,l,m+1,\dots,m+n})^t \\ M_{1,\dots,l,1,\dots,m} & M_{1,\dots,l,m+1,\dots,m+n} & 0 & 0 \\ M_{1,\dots,l,1,\dots,m} & M_{1,\dots,l,m+1,\dots,m+n} & 0 & 0 \end{bmatrix} \begin{bmatrix} x^t \\ y^t \\ \mu^t \\ \nu^t \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ b^t \\ b^t \end{bmatrix},$$

$$0 = [\mathrm{x} \quad \mathrm{y} \quad \mu \quad \nu] \begin{bmatrix} u_1^t \\ u_2^t \\ v^t \\ v^t \end{bmatrix},$$

$$0 \le x, y, \mu, \nu, u_1, u_2, v.$$

LCPs are quite common and can be solved, even though it might be necessary to perturb the problem to avoid degeneration. Hence, we can solve a PBG by means of the above lemma.

Additionally using this characterization, we consider the shape of the set of equilibria.

Theorem 5.3. Let $\langle P, A, B\rangle$ be a PBG. The set of equilibria $E_P(A, B)$ is the finite union of convex polytopes.

This theorem follows directly from the fact that the set of equilibria $E_P(A, B)$ is equal to the set of solutions to the LCP; Jansen (Ref. 4) has shown this set to be a finite union of polytopes.

Jansen (Ref. 5) has also investigated the structure of the set of equilibria found in bimatrix games, where the set $E(A, B)$ is the union of maximal Nash subsets, which are convex polytopes. The Nash subsets have the property that all strategy pairs in these sets are interchangeable. However, for polytope bimatrix games, the meaning and usefulness of the concept of maximal Nash subsets is not clear, as the following example shows.

Example 5.1. Consider the bimatrix game defined by

$$A=\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix},\qquad B=\begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix}.$$

The strategy space $S_m \times S_n$ is restricted to

$$P := \{(x, y)\in S_m \times S_n \,|\, x_1 + y_2 \leq 1\}.$$

This is a restricted game with

$$R_1 := xAy^t,\qquad R_2 := xBy^t$$

continuous and concave on P, and P convex and compact. The set of equilibria is

$$E_P(A, B) = \{((0, 1), (1, 0))\} \cup \{(x, y)\,|\, x_1 + y_2 = 1\}.$$

Figure 2 shows the original set $S_m \times S_n$. The restricted strategy set P is shown shaded and the equilibria are circled. The strategy pair $((1, 0), (0, 1))$ is circled, even though it is not even in P, because it is an equilibrium point of the original bimatrix game (A, B).

By using the results from Section 4, one can see that

$$((0, 1), (0, 1))\in E_P(A, B),\qquad ((0, 1), (1, 0))\in E_P(A, B).$$

These two equilibria are $E_P(A, B)$-interchangeable, meaning that we can construct a Nash subset

$$S = \{((0, 1), (0, 1)), ((0, 1), (1, 0))\}.$$

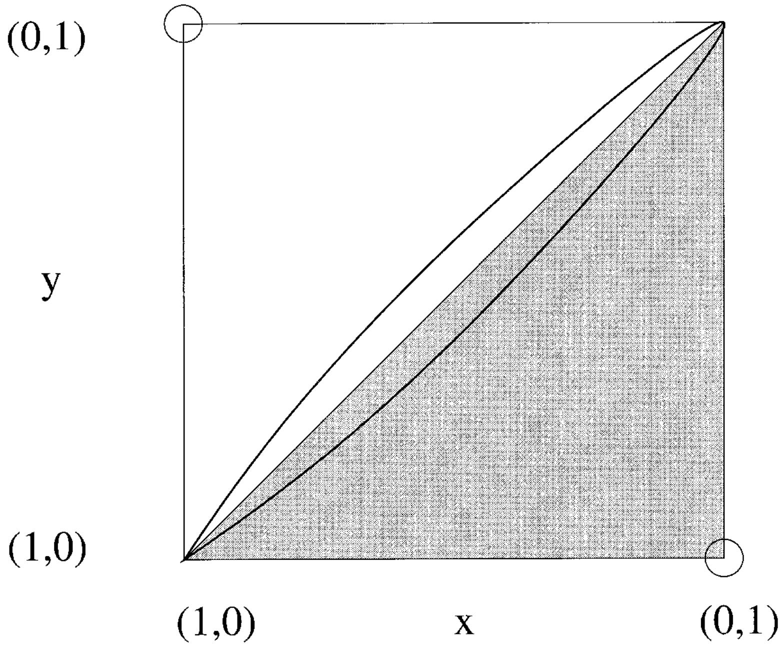

Fig. 2. Example 5.1.

Considering the polytope, one can also see that there exist no other equilibria which are interchangeable with elements of S; hence, S is a maximal Nash subset. However, as one can easily see, this maximal Nash subset is not convex and

$$\text{conv hull}\{((0,1),(0,1)),((0,1),(1,0))\} \not\subseteq E_P(A,B).$$

Convexity is one of the most important characteristic of maximal Nash subsets; clearly, it no longer holds for the polytope bimatrix game.

6. Refinements of the Equilibrium Concept

In this section, we extend refinements to the equilibrium concept, introduced by Harsanyi (Ref. 6), Selten (Ref. 7), and Myerson (Ref. 8) for bimatrix games, to polytope bimatrix games. All results are generalizations, in the sense that, in the special case of a PBG where the polytope is itself $S_m \times S_n$, the definitions are equivalent with the standard definitions for bimatrix games.

Definition 6.1. A strategy pair $(x, y) \in P$ is called undominated if, for all $\bar{x} \in X(y)$ and for all $\bar{y} \in Y(x)$,

$$\bar{x}A \geq xA \text{ implies } \bar{x}A = xA,$$

$$B\bar{y}^t \geq By^t \text{ implies } B\bar{y}^t = By^t.$$

Undominated equilibria of bimatrix games have been investigated by Borm et al. (Ref. 9).

Definition 6.2. Let $\langle P, A, B \rangle$ be a PBG, and let P be characterized as in Lemma 5.2. For $\epsilon \in \mathbb{R}^+$, the ϵ-perturbed PBG is defined as $\langle P_\epsilon, A, B \rangle$, where

$$P_\epsilon = \left\{ (x, y) \in \mathbb{R}^{m+n} \,\middle|\, M \begin{bmatrix} x^t \\ y^t \end{bmatrix} \leq \begin{bmatrix} b_e \\ b_p - \epsilon \end{bmatrix} \text{and } x, y \geq \epsilon \right\}.$$

Then, this is a simple ϵ-contraction of the polytope P; one should note that, for sufficiently small ϵ, the polytope structure remains the same, i.e., it has the same number of faces, etcetera.

Having so defined the ϵ-perturbed polytopes, we define a perfect equilibrium in such a way that it conforms with the perfectness concept proposed by Selten in Ref. 7.

Definition 6.3. Let $\langle P, A, B \rangle$ be a PBG, with $\dim(P) = m + n - 2$ and $\operatorname{relint}(P) \neq \emptyset$. The pair $(x, y) \in E_P(A, B)$ is called a perfect equilibrium if there exists a sequence $(\epsilon_k)_k \to 0$ and also a sequence $(x_k, y_k)_{k \in \mathbb{N}}$, with $(x_k, y_k) \in E_{P_{\epsilon_k}}(A, B)$ for all $k \in \mathbb{N}$, such that $(x_k, y_k)_k \to (x, y)$.

Restricted to the special case where the polytope is the original strategy set, this defines perfect equilibria. We pose the question of whether perfect equilibria always exist for PBG as they do for bimatrix games. To answer this, we need the following property.

Definition 6.4. A sequence of sets $(V_n)_{n \in \mathbb{N}}$, where $V_n \subset \mathbb{R}^n$ for some $n \in \mathbb{N}$, is said to converge to $V \subset \mathbb{R}^n$ $(V_n \to V)$ if:

(i) for all $(v_n)_{n \in \mathbb{N}}$, $v_n \in V_n$, with $v_n \to_n v$, we have $v \in V$;
(ii) for all $v \in V$, there exists $(v_n)_n$, $v_n \in V_n$ such that $v_n \to_n v$.

Lemma 6.1. Let $(x_k, y_k) \in P_{\epsilon_k}$ be a sequence with $\epsilon_k \to_k 0$ and $(x_k, y_k) \to_k (x, y)$. Let

$$X_{\epsilon_k}(y_k) := \{x \in S_m \,|\, (x, y_k) \in P_{\epsilon_k}\},$$

$$Y_{\epsilon_k}(x_k) := \{y \in S_n \,|\, (x_k, y) \in P_{\epsilon_k}\}.$$

Then, $X_{\epsilon_k}(y_k) \to_k X(y)$ and $Y_{\epsilon_k}(x_k) \to_k Y(x)$.

The proof is a simple application of the preceding definition. We now show the existence of perfect equilibria.

Theorem 6.1. For every PBG$\langle P, A, B\rangle$, where relint$(P) \neq \emptyset$, there exists a perfect equilibrium.

Proof. Since relint$(P) \neq \emptyset$, there exists ϵ_0 such that, for all $\epsilon < \epsilon_0$, the game $\langle P_\epsilon, A, B\rangle$ is a polytope bimatrix game. Because every PBG has equilibria and the set P is compact, it follows that there exists a sequence $(x_k, y_k)_{k\in\mathbb{N}}$, with $(x_k, y_k) \in E_{P_{\epsilon_k}}(A, B)$ for all $k \in \mathbb{N}$, such that

$$(x_k, y_k) \to_{k\to\infty} (x, y), \qquad \text{for some } (x, y) \in P.$$

All that is left to show is that $x \in O_I(y)$. Because

$$X_{\epsilon_k}(y_k) \to_k X(y)$$

and

$$x_k = \arg\max_{x\in X_{\epsilon_k}(y_k)} x(Ay^t),$$

it follows from the continuity of $x(Ay^t)$ that

$$x = \arg\max_{x\in X(y)} x(Ay^t).$$

Thus, it follows that $x \in O_I(y)$ and the same argument can be applied to $y \in O_{II}(x)$. □

As Van Damme (Ref. 10) has shown that, for bimatrix games, perfect and undominated equilibria are equivalent, we show that, for polytope bimatrix games, perfect equilibria imply undominated equilibria, but not the converse.

Definition 6.5. For a set $V \subset \mathbb{R}^n$, define the subset of undominated points UND(V) as follows:

$$\text{UND}(V) := \{v \in V \mid \text{there exists no } \bar{v} \in V \text{ such that } v \leq \bar{v} \text{ and } v_j < \bar{v}_j \text{ for some } j\}.$$

Lemma 6.2. Let $V_n \subset \mathbb{R}^n$ be a sequence of polytopes such that $V_n \to V$ and there exists $n_p \in \mathbb{N}$ such that, for all $n \geq n_p$, the supporting hyperplanes of V_n are parallel to those of V. Then, UND$(V_n) \to_n$ UND(V).

Continuity arises from the fact that, at some ϵ, all restrictions are parallel to the original prototype.

Theorem 6.2. Let $\langle P, A, B\rangle$ be a PBG, and let $\dim(P) = m + n - 2$. If (x, y) is a perfect equilibrium, then (x, y) is undominated.

Proof. Let (x_k, y_k) be the sequence of ϵ_k-equilibria, $(x_k, y_k) \in E_{P_{\epsilon_k}}(A, B)$ for all $k \in \mathbb{N}$, that converges to (x, y). Then, it follows that, for all $k \in \mathbb{N}$, $x_k A$ is undominated on $\{xA \,|\, x \in X_{\epsilon_k}(y_k)\}$, since $y_k > 0$ is a completely mixed strategy under the assumption of full dimension.

From this and the fact that $X_{\epsilon_k}(y_k) \to_k X(y)$, it follows with the continuity of the linear map that

$$A(X_{\epsilon_k}(y_k)) \to_k A(X(y)),$$

and so by Lemma 6.2 it follows that

$$\mathrm{UND}(A(X_{\epsilon_k}(y_k))) \to_k \mathrm{UND}(A(X(y))).$$

Since

$$x_k A \in \mathrm{UND}(A(X_{\epsilon_k}(y_k))) \quad \text{and} \quad x_k \to x,$$

it follows that x is undominated on $X(y)$. □

The converse does not hold in the case of polytope bimatrix games, which we show in the following example.

Example 6.1. Consider the bimatrix game defined by

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

The strategy space $S_m \times S_n$ is restricted to

$$P := \{(x, y) \in S_m \times S_n \,|\, x_1 + y_2 \leq 1, x_2 \leq 1 - (1/2)y_2, y_2 \geq (1/2)x_1\}.$$

This defines a polytope bimatrix game $\langle P, A, B\rangle$.

By using the results in Section 4, it is clear that the set of all equilibria is

$$E_P(A, B) = \{((0, 1), (1, 0))\}$$
$$\cup \{(x, y) \,|\, x_1 + y_2 = 1, x_2 \leq 1 - (1/2)y_2, y_2 \geq (1/2)x_1\},$$

which is depicted in Fig. 3. In this picture the equilibria are circled and they coincide with the undominated points. Especially note that the point $((0, 1), (1,0))$ is undominated, mainly because $X(y)$ and $Y(x)$ include only one point. However, this equilibrium is not perfect, as any perturbation of the polytope yields equilibria in the upper left side of the polytope.

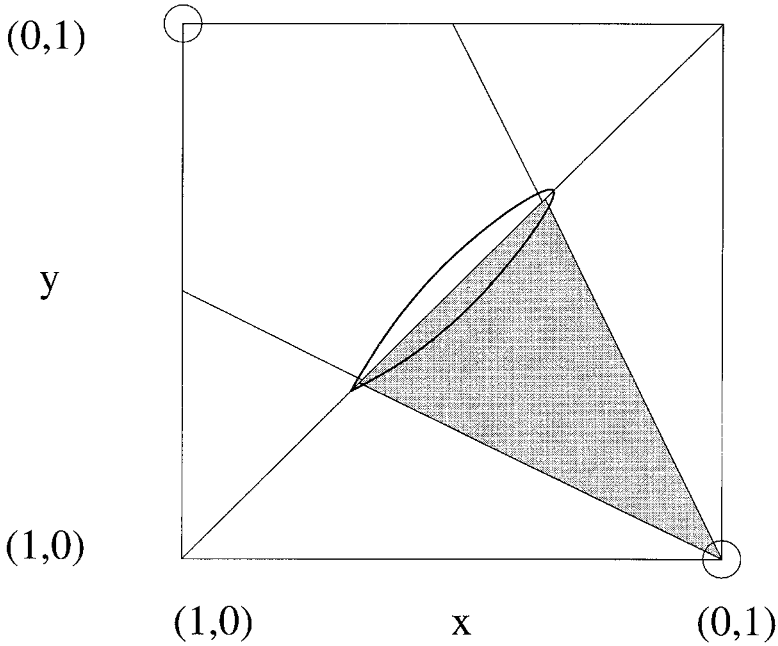

Fig. 3. Example 6.1.

Finally, we look at a further refinement to the equilibrium concept, that of proper equilibria. For bimatrix games, this was proposed by Myerson (Ref. 8); we extend this idea to polytope bimatrix games.

Definition 6.6. Let $\langle P, A, B\rangle$ be a PBG, $(x, y) \in \operatorname{relint}(P)$, $X(y) = \operatorname{conv\,hull}\{x_1, \ldots, x_k\}$, $Y(x) = \operatorname{conv\,hull}\{y_1, \ldots, y_l\}$, and $\epsilon > 0$. The pair (x, y) is called ϵ-proper equilibrium if the conditions below are satisfied:

(i) for any set $\{i_1, \ldots, i_q\}$ and index r with

$$x_{i_m} A y^t < x_r A y^t, \qquad \text{for all } m \in \{1, \ldots, q\},,$$

there exist $\mu_1, \ldots, \mu_k$ with $x = \sum_{i=1}^{k} \mu_i x_i$, $\sum_{m=1}^{q} \mu_{i_m} \leq \epsilon \mu_r$;

(ii) for any set $\{j_1, \ldots, j_p\}$ and index s with

$$x B y_{j_n}^t < x B y_s^t, \qquad \text{for all } n \in \{1, \ldots, p\},$$

there exist $\nu_1, \ldots, \nu_l$ with $y = \sum_{j=1}^{l} \nu_j y_j$, $\sum_{n=1}^{p} \nu_{j_n} \leq \epsilon \nu_s$.

Definition 6.7. Let $\langle P, A, B\rangle$ be a PBG, with $\dim(P) = m + n - 2$ and $\operatorname{relint}(P) \neq \emptyset$. The pair $(x, y) \in E_P(A, B)$ is called a proper equilibrium if there exists a sequence $(\epsilon_k)_k \to 0$ and a sequence $(x_k, y_k)_{k\in\mathbb{N}}$, with (x_k, y_k) an ϵ_k-proper equilibrium for all $k\in\mathbb{N}$, such that $(x_k, y_k)_k \to (x, y)$.

Theorem 6.3. For every PBG, $\langle P, A, B\rangle$, where $\mathrm{relint}(P)\neq\emptyset$, there exists a proper equilibrium.

Proof. It suffices to show that, for ϵ small enough, there exist ϵ-proper equilibria. To prove this, we first define the mapping

$$PR_1^\epsilon(y) := \Big[x\in X(y) \,|\, x_{i_m}Ay^t < x_r Ay^t, \text{ for all } m\in\{1, \dots, q\},$$
$$\text{implies that there exist } \mu_1, \dots, \mu_k \text{ with}$$
$$x = \sum_{i=1}^{k} \mu_i x_i,\ \sum_{m=1}^{q} \mu_{i_m} \leq \epsilon\mu_r \text{ and}$$
$$\mu_i \geq [\epsilon/(1+\epsilon)]^{k+1}, \text{ for all } i\in\{1, \dots, k\}\Big].$$

Then, we show that this mapping has properties (i) to (iii) below.

(i) $PR_1^\epsilon(y)$ is convex and compact. Let $x', x''\in PR_1^\epsilon(y)$. Then, for all $\{i_1, \dots, i_o\}\subset\{1, \dots, k\}$ and $t\in\{1, \dots, k\}$, if

$$x_{i_m}Ay^t < x_r Ay^t, \qquad \text{for all } m\in\{1, \dots, q\},$$

there exist $\mu_1', \dots, \mu_k'$ and $\mu_1'', \dots, \mu_k''$ with

$$x' = \sum_{i=1}^{k} \mu_i' x_i, \qquad \sum_{m=1}^{q} \mu_{i_m}' \leq \epsilon\mu_r',$$

$$\mu_i' \geq [\epsilon/(1+\epsilon)]^{k+1}, \qquad \text{for all } i\in\{1, \dots, k\},$$

$$x'' = \sum_{i=1}^{k} \mu_i'' x_i, \qquad \sum_{m=1}^{q} \mu_{i_m}'' \leq \epsilon\mu_r'',$$

$$\mu_i'' \geq [\epsilon/(1+\epsilon)]^{k+1}, \qquad \text{for all } i\in\{1, \dots, k\}.$$

Then,

$$\begin{aligned} &\lambda(x') + (1-\lambda)x'' \\ &= \sum \lambda\mu_i' x_i + \sum (1-\lambda)\mu_i'' x_i \\ &= \sum \bar{\mu}_i x_i, \end{aligned}$$

with

$$\bar{\mu}_i \geq [\epsilon/(1+\epsilon)]^{k+1}, \qquad \text{for all } i\in\{1, \dots, k\}.$$

Now,

$$\begin{aligned}
&\sum_{m=1}^{q} [\lambda\mu'_{i_m} + (1-\lambda)\mu''_{i_m}] \\
&= \sum_{m=1}^{q} \lambda\mu'_{i_m} + \sum_{m=1}^{q} (1-\lambda)\mu''_{i_m} \\
&= \lambda \sum_{m=1}^{q} \mu'_{i_m} + (1-\lambda) \sum_{m=1}^{q} \mu''_{i_m} \\
&\le \epsilon[\lambda\mu'_r + (1-\lambda)\mu''_r].
\end{aligned}$$

So, it follows that

$$\lambda(x') + (1-\lambda)x'' \in PR_1^\epsilon(y);$$

hence, $PR_1^\epsilon(y)$ is convex and compact.

(ii) $PR_1^\epsilon(y)$ is nonempty. First, we order x_i with respect to the payoff so that

$$x_{i_1}Ay^t \le x_{i_2}Ay^t \le \cdots \le x_{i_k}Ay^t,$$

and partition $\{i_1, \ldots, i_k\}$ into subsets $T_1, \ldots, T_s$ such that

for any $i, j \in T_k$, we have $x_iAy^t = x_jAy^t$,

for any $m < n$, $i \in T_m$, $j \in T_n$, we have $x_iAy^t < x_jAy^t$.

Now, we can construct $\mu_1, \ldots, \mu_k$ in $s-1$ steps.

Step 1. For all $i \in T_s$, let $\mu_i := 1/|T_s|$.
Step 2. From one $i \in T_s$, subtract $a_1 := [\epsilon/(1+\epsilon)]\mu_i$ and define $\mu_i := \mathrm{a}_1/|T_{s-1}|$, for all $i \in T_{s-1}$.
Step 3. Repeat Step 2 for T_{s-1} and T_{s-2}.
. . .
Step $s-1$. From one $i \in T_2$, subtract $a_{s-1} := [\epsilon/(1+\epsilon)]\mu_i$ and define $\mu_i := a_{s-1}/|T_1|$, for all $i \in T_1$.

Easy calculation shows that, for ϵ small enough such a $\mu_1, \ldots, \mu_k$ fulfills the restriction.

(iii) $PR_1^\epsilon(y)$ is upper semicontinuous in y. Let $y^n \to y$ and $x^n \to x$ with $x^n \in PR_1^\epsilon(y^n)$. Because $X(y^n) \to X(y)$, it follows that

$$\mathrm{extr}(X(y^n)) \to \mathrm{extr}(X(y)).$$

Since both $\mathrm{extr}(X(y^n))$ and $\mathrm{extr}(X(y))$ are finite sets, we can partition their elements

$$K(x_i) := \{(x^n)_n \,|\, x^n \in \mathrm{extr}(X(y^n)) \text{ and } x^n \to x_i\}$$

further,

$$K_n(x_i) := \left[\bigcup_{(x^n)_n \in K(x_i)} \{x^1, \dots, \} \right] \cap X(y^n).$$

For some $\{i_1, \dots, i_q\}$ and r,

$$x_{i_m} A y^t < x_r A y^t, \qquad \text{for all } m \in \{1, \dots, q\};$$

then, it follows by continuity that, for n large enough,

$$xA(y^n)^t < \bar{x}A(y^n)^t, \qquad \text{for all } m \in \{1, \dots, q\},$$
$$\text{for all } x \in K_n(x_{i_m}), \quad \text{and } \bar{x} \in K_n(x_r).$$

So, since $x^n \in PR_1^\epsilon(y^n)$, it follows that there exist expressions

$$x^n = \sum_{i=1}^{\Sigma_j |K_n(x_j)|} \mu_i^n x_i^n, \qquad \sum_{i \in \{k | x_k \in \cup_{m=1}^q K_n(x_{k_m})\}} \mu_i^n \leq \epsilon \mu_r^n.$$

Now, it follows that

$$x = \lim_{n \to \infty} x^n = \lim_{n \to \infty} \sum_{i=1}^{\Sigma_j |K_n(x_j)|} \mu_i^n x_i^n.$$

Since $X(y)$ is continuous, it follows that

$$\sum_j |K_n(x_j)| = |\text{extr}(X(y^n))|$$

is constant for n large enough and can only decrease at y,

$$x = \lim_{n \to \infty} x^n = \lim_{n \to \infty} \sum_{i=1}^{\Sigma_j |K_n(x_j)|} \mu_i^n x_i^n = \sum_{i=1}^{\Sigma_j |K_n(x_j)|} \lim \mu_k^n \lim x_k^n.$$

Now, we can see that

$$\sum_{m=1}^q \mu_{i_m} = \sum_{i \in \{k | x_k \in \cup_{m=1}^q K_n(x_{k_m})\}} \lim \mu_i^n$$
$$= \lim \sum_{i \in \{k | x_k \in \cup_{m=1}^q K_n(x_{k_m})\}} \mu_i^n \leq \epsilon \lim \mu_r^n = \epsilon \mu_r,$$

which proves that $x \in PR_1^\epsilon(y)$, and so the upper semicontinuity is proven.

In the same way, we can define a mapping $PR_2^\epsilon(x)$. Since both mappings are compact-valued, convex-valued, and upper semicontinuous, we can use the proof of the existence theorem (Section 3) in exactly the same way to yield the existence of ϵ-proper equilibria. □

7. Conclusions

In this article, we have studied a generalization of the classical bimatrix game. The distinction is that the strategies of the players are not independent. While this proved to be a more complex game, it was shown that most of the results could be extended. The key lies in two linear programming problems, which are solved simultaneously, since these describe the set of equilibria and give an elegant way to compute an equilibrium.

References

1. NASH, J., *Noncooperative Games*, Annals of Mathematics, Vol. 54, pp. 286–295, 1951.
2. KAKUTANI, S., *A Generalization of Brouwer's Fixed-Point Theorem*, Duke Mathematical Journal, Vol. 8, pp. 457–459, 1941.
3. LUENBERGER, D. G., *Linear and Nonlinear Programming*, Addison-Wesley, Reading, Massachusetts, 1973.
4. JANSEN, M. J. M., *On the Structure of the Solution Set of a Linear Complementarity Problem*, Cahiers du Centre d'Etudes de Recherche Operationelle, Vol. 25, pp. 41–48, 1983.
5. JANSEN, M. J. M., *Maximal Nash Subsets for Bimatrix Games*, Naval Research Logistics Quarterly, Vol. 28, pp. 147–152, 1981.
6. HARSANYI, J. C., *A Solution Concept for N-Person Noncooperative Games*, International Journal of Game Theory, Vol. 5, pp. 211–225, 1976.
7. SELTEN, R., *Reexamination of the Perfectness Concept for Equilibrium Points in Extensive Games*, International Journal of Game Theory, Vol. 4, pp. 25–55, 1975.
8. MYERSON, R. B., *Refinements of the Nash Equilibrium Concept*, International Journal of Game Theory, Vol. 7, pp. 73–80, 1978.
9. BORM, P., JANSEN, M., POTTERS, J., and TIJS, S., *On the Structure of the Set of Undominated Equilibria in Bimatrix Games*, Report 8819, Department of Mathematics, Nijmegen University, Nijmegen, Netherlands, 1988.
10. VAN DAMME, E., *Stability and Perfection of Nash Equilibria*, Springer Verlag, Berlin, Germany, 1987.