Abstract. Inconsistent knowledge-bases can entail useful conclusions when using the three-valued semantics of the paraconsistent logic LP. However, the set of conclusions entailed by a consistent knowledge-base under the three-valued semantics is smaller than the set of conclusions entailed by the knowledge-base under a two-valued semantics. Preferring conflict-minimal interpretations of the logic LP; i.e., LPm, reduces the gap between these two sets of conclusions.

Preferring conflict-minimal interpretations introduces non-monotonicity. To handle the non-monotonicity, this paper proposes an assumption-based argumentation system. Assumptions needed to close branches of a semantic tableaux form the arguments. Stable extensions of the set of derived arguments correspond to conflict minimal interpretations and conclusions entailed by all conflict-minimal interpretations are supported by arguments in all stable extensions.

1 Introduction

In an open and distributed environment such as the internet, knowledge and information originating from different sources need not be consistent. As a result, when using a standard two-valued semantics, no useful conclusions can be derived. Everything is entailed because the set of two-valued interpretations is empty. Resolving the inconsistencies is often not an option in an open and distributed environment. Therefore, methods that allow us to derive useful conclusions in the presence of inconsistencies are preferred.

One possibility to draw useful conclusions from inconsistent knowledge and information is by using a paraconsistent logic. In this paper we focus on the paraconsistent logic LP [17]. LP, which stands for Logic of Paradox, is based on a three-valued semantics. The LP-semantics assigns TRUE, FALSE or CONFLICT to each proposition. It differs from Belnap’s semantics [2] in not allowing the truth-value unknown.

An important advantage of the paraconsistent logic LP is that the entailment relation is monotonic. A disadvantage is that consistent knowledge and information entails fewer conclusions when using the three-valued semantics, than when using the two-valued semantics. Priest [17, 18] proposed the use of conflict-minimal interpretations in LP to reduce the gap between the sets of conclusions entailed by the two semantics. The focus on conflict-minimal interpretations makes the resulting logic LPm non-monotonic [18].

In this paper we present an argumentation system for conclusions entailed in LPm. We start from a semantic tableaux method for LP and Belnap’s logic, proposed by Bloesch [5]. The tableaux is used for deriving all conclusions entailed under the LP-semantics. If a tableaux cannot be closed, the desired conclusion may still hold in all conflict-minimal interpretations. The open tableaux enables us to identify assumptions about conflict-minimality. These assumptions are used to construct an assumption-based argumentation system, which supports conclusions entailed by all conflict minimal interpretations.

The remainder of the paper is organized as follows. The next section reviews the paraconsistent logics LP and LPm, and Bloesch’s semantic tableaux method. Section 3 describes how Bloesch’s semantic tableaux method can be used to determine arguments for conclusions supported by conflict-minimal interpretations of LPm. Subsequently, in Section 4, an outline of the correctness and completeness proof of the described approach is given. Section 5 describes some related work. The last section concludes the paper.

2 LP and LPm

In the paper we will focus on the paraconsistent logic LP and on the logic LPm, which minimizes the conflicts in the interpretations of LP. The logic LP is a three-valued logic with the truth-values TRUE, FALSE and CONFLICT. We can view the truth-values of LP as sets of truth-values of the classical Tarski semantics: \{t\}, \{f\}, and \{t, f\}. Hence, instead of a two-valued interpretation \(I : P \rightarrow \{t, f\}\) assigning \(t\) or \(f\) to atomic propositions in \(P\), we assign a set of classical truth-values: \(I : P \rightarrow (2^{\{t, f\}} - \emptyset)\). The language \(L\) of all propositions is recursively defined starting from the set of atomic propositions \(P\) using the logical operators \(\neg\), \(\land\) and \(\lor\). The truth-values of these propositions are determined by the extended interpretation function \(I^* : L \rightarrow (2^{\{t, f\}} - \emptyset)\). This extended interpretation function is recursive defined by the following truth-tables:

\[
\begin{array}{ccc}
\neg & \{t\} & \{f\} \\
& \{f\} & \{t\} \\
& \{t, f\} & \{t, f\} \\
\land & \{t\} & \{f\} & \{t, f\} \\
& \{f\} & \{f\} & \{f\} \\
& \{t, f\} & \{t, f\} & \{t, f\} \\
\lor & \{t\} & \{f\} & \{t\} \\
& \{f\} & \{f\} & \{t\} \\
& \{t, f\} & \{t, f\} & \{t, f\} \\
\end{array}
\]

The relation between the truth-value assignments and the three-valued entailment relation is given by: \(I \models \varphi\) iff \(I^* (\varphi)\)

Note that we get Belnap’s four-valued logic if we also allow the empty set of truth-values [2]. A disadvantage of Belnap’s logic compared to LP, is that tautologies need not hold because the truth-value
of some atomic proposition is unknown. Although the result presented in the paper also apply to Belnap’s logic, because of this disadvantage, we will focus on LP in the paper.

Bloesch [5] proposed a semantic tableaux method for both LP and Belnap’s logic. We will use this semantic tableaux method because it enables us to handle conflict minimal interpretations. Bloesch’s semantic tableaux method associates a label with every proposition in the tableaux. Possible labels are: \( T \) (at least true), \( F \) (at least false), or their complements \( \overline{T} \) and \( \overline{F} \), respectively. So, \( T \varphi \) corresponds to \( t \in I(\varphi) \), \( \overline{T} \varphi \) corresponds to \( t \notin I(\varphi) \), \( F \varphi \) corresponds to \( f \in I(\varphi) \), and \( \overline{F} \varphi \) corresponds to \( f \notin I(\varphi) \).

Although we do not need it in the semantic tableaux, we also make use of \( C \varphi \) and \( C \overline{\varphi} \), which corresponds semantically with \( I(\varphi) = \{ t, f \} \) and \( I(\varphi) \neq \{ t, f \} \), respectively. So, \( C \varphi \) is equivalent to: \( T \varphi \) and \( F \varphi \); and \( C \overline{\varphi} \) is equivalent to: \( T \overline{\varphi} \) or \( F \overline{\varphi} \).

To prove that \( \Sigma \models \varphi \) using Bloesch’s tableaux method [5], we have to show that a tableaux with root \( \Gamma = \{ \varphi \} \) contains fewer conflicts than the interpretations \( I \) of \( \varphi \).

The conflict-minimal entailment of a proposition by a set of propositions can now be defined.

**Definition 2** Let \( I_1 \) be a three-valued interpretation and let \( \Sigma \) be a set of propositions.

\( I_1 \) is a conflict minimal interpretation of \( \Sigma \), denoted by \( I_1 \models \Sigma \), iff \( I_1 \models \Sigma \) and for no interpretation \( I_2 \) such that \( I_2 \models \Sigma \), \( I_2 \models \Sigma \) holds.

In Example 2, \( I_1 \) is the only conflict-minimal interpretation.

The conflict-minimal entailment of a proposition by a set of propositions can now be defined.

**Definition 3** Let \( \Sigma \subseteq \mathcal{L} \) be a set of propositions and let \( \varphi \in \mathcal{L} \) be a proposition.

\( \Sigma \) entails conflict-minimally the proposition \( \varphi \), denoted by \( \Sigma \models \varphi \), iff for every interpretation \( I \), if \( I \models \Sigma \), then \( I \models \varphi \).

The conflict-minimal interpretations in Example 2 entail the conclusion \( q \).

### 3 Arguments for conclusions supported by conflict minimal interpretations

The conflict-minimal interpretations of a knowledge base entail more useful conclusions than the three-valued interpretations of the knowledge base. Unfortunately, focusing on conclusions supported by conflict-minimal interpretations makes the reasoning process non-monotonic. Adding the proposition \( \neg q \) to the set of propositions in Example 2 eliminates interpretations \( I_1 \) and \( I_3 \), which includes the only conflict-minimal interpretation \( I_1 \). The interpretations \( I_2 \) and \( I_4 \) are the new conflict-minimal interpretations. Unlike the original conflict-minimal interpretation \( I_1 \), the new conflict-minimal interpretations \( I_2 \) and \( I_4 \) do not entail \( q \).

Deriving conclusions supported by the conflict-minimal interpretations is problematic because of the non-monotonicity. The modern way to deal with non-monotonicity is by giving an argument supporting a conclusion and subsequently verifying whether there are no counter-arguments [10]. Here we will follow this argumentation-based approach.

We propose an approach for deriving arguments that uses the semantic tableaux method for our paraconsistent logic as a starting point. The approach is based on the observation that an interpretation satisfying the root of a semantic tableaux will also satisfy one of the leafs. Now suppose that the only leafs of a tableaux that are not closed; i.e., leafs in which we do not have “\( \overline{T} \alpha \) and \( \overline{T} \alpha \)” or “\( \overline{F} \alpha \) and \( \overline{F} \alpha \)” or “\( \overline{T} \alpha \) and \( \overline{F} \alpha \)”, are leafs in which “\( \overline{T} \alpha \) and \( \overline{F} \alpha \)” holds for some propositions \( \alpha \). So, in every open branch of the tableaux, \( \overline{T} \alpha \) holds for some proposition \( \alpha \). If we can assume that there are no conflicts w.r.t. each proposition \( \alpha \) in the conflict-minimal interpretations, then we can also close the open branches. The set of assumptions \( \overline{\Sigma} \alpha \), equivalent to \( \overline{\overline{\overline{T} \alpha} \overline{\overline{T} \alpha}} \), that we need to close the open branches, will be used as the argument for the conclusion supported by the semantic tableaux.

A branch that can be closed assuming that the conflict-minimal interpretations contain no conflict with respect to a proposition \( \alpha \); i.e., assuming \( \overline{\overline{\overline{\overline{\overline{T} \alpha} \overline{\overline{T} \alpha}}}} \), will be called a weakly closed branch. We will call a tableaux weakly closed if some branches are weakly closed and
all other branches are closed. If we can (weakly) close a tableaux for
\(\Gamma = \{\exists \sigma \mid \sigma \in \Sigma\} \cup \exists \phi\), we consider the set of the assumptions \(\exists \alpha\)
needed to weakly close the tableaux, to be the argument supporting
\(\exists \mid \leq \phi\). Example 3 gives an illustration.

**Example 3** Let \(\Sigma = \{-p, p \lor q\}\) be a set of propositions. To verify
whether \(q\) holds, we may construct the following tableaux:

\[
\begin{array}{c}
T \neg p \\
T p \lor q \\
\exists \neg p \\
F p \\
T p \\
F q \\
\otimes \{p\} \\
\otimes \{q\} \\
\end{array}
\]

Only the left branch is weakly closed in this tableaux. We assume
that the proposition \(p\) will not be assigned CONFLICT in any conflict-
minimal interpretation. That is, we assume that \(\exists p\) holds.

In the following definition of an argument, we consider arguments
for \(\exists p\) and \(F \phi\).

**Definition 4** Let \(\Sigma\) be a set of propositions and let \(\phi\) a proposi-
tion. Moreover, let \(T\) be a (weakly) closed semantic tableaux with
root \(\Gamma = \{\exists \sigma \mid \sigma \in \Sigma\} \cup \exists \phi\) and \(L \in \{T, F\}\). Finally, let
\(\{\exists \alpha_1, \ldots, \exists \alpha_k\}\) be the set of assumptions on which the closures
of weakly closed branches are based.

Then \(A = \{(\exists \alpha_1, \ldots, \exists \alpha_k), \exists \phi\}\) is an argument for \(\exists \phi\).

The next step is to verify whether the assumptions: \(\exists \alpha\) are valid.
If one of the assumptions does not hold, we have a counter-argument
for our argument supporting \(\Sigma \mid \leq \phi\). To verify the correctness
of an assumption, we add the assumption to \(\Sigma\). Since an assumption
\(\exists \alpha\) is equivalent to: \(\neg \exists \alpha \lor \exists \neg \alpha\), we can consider \(\exists \alpha\) and \(\exists \neg \alpha\) sepa-
rately. Example 4 gives an illustration for the assumption \(\exists p\) used in
Example 3.

**Example 4** Let \(\Sigma = \{-p, p \lor q\}\) be a set of propositions. To verify
whether the assumption \(\exists p\) holds in every conflict minimal inter-
pretation, we may construct a tableaux assuming \(\exists p\) and a tableaux
assuming \(\exists \neg p\):

\[
\begin{array}{c}
T \neg p \\
T p \lor q \\
\exists \neg p \\
F p \\
T p \\
F q \\
\otimes \{p\} \\
\otimes \{q\} \\
\end{array}
\]

The right branch of the first tableaux cannot be closed. Therefore,
the assumption \(\exists p\) is valid, implying that the assumption \(\exists \neg p\) is also
valid. Hence, there exists no counter-argument.

Since the validity of assumptions must be verified with respect to
conflict-minimal interpretations, assumptions may also be used in the
counter-arguments. This implies that we must have to verify whether
there exists a counter-argument for a counter-argument. Example 5
gives an illustration.

**Example 5** Let \(\Sigma = \{-p, p \lor q, \neg q \lor r, \neg r\}\) be a set of propositions.
To verify whether \(q\) holds, we may construct the following tableaux:

\[
\begin{array}{c}
T \neg p \\
T p \lor q \\
\exists \neg p \\
F p \\
T p \\
\neg q \lor r \\
\otimes \{q\} \\
\otimes \{r\} \\
\end{array}
\]

This weakly closed tableaux implies the argument \(A_0 = \{(\exists \neg p), \exists q\}\). Next we have to verify whether there exists a counter-
argument for \(A_0\). To verify the existence of a counter-argument, we can
construct the following two tableaux:

\[
\begin{array}{c}
T \neg p \\
T p \lor q \\
T q \\
\otimes \{q\} \\
\otimes \{r\} \\
\end{array}
\]

Both tableaux are (weakly)-closed, and therefore form the counter-
argument \(A_1 = \{(\exists p, \exists r), \exists q\}\). We say that the argument \(A_1\) attac-
tacks the argument \(A_0\) because the former is a counter-argument of
the latter.

The two tableaux forming the counter-argument \(A_1\) are closed under
the assumptions: \(\exists p\) and \(\exists r\). So, \(A_1\) is a valid argument if there exists
no valid counter-argument for \(q\) and no counter-argument
for \(\exists r\).

Note that the argument \(A_1\) implies that the set of assumptions
\(\{\exists p, \exists r, \exists q\}\) together with \(\{\exists \sigma \mid \sigma \in \Sigma\}\) is not satisfi-
table. Therefore, \(A_1\) implies two other arguments, namely: \(A_2 = \{(\exists p, \exists r), \exists q\}\) and \(A_3 = \{(\exists p, \exists q), \exists r\}\). Clearly, the argument
\(A_1\) is a counter-argument of \(A_2\) and \(A_3\). \(A_2\) is a counter-argument of \(A_1\) and \(A_3\), and \(A_3\) is a counter-arguments of \(A_1\) and \(A_2\). No other
counter-arguments can be identified. Figure 1 shows arguments. The
arrows denote which arguments is a counter-argument of another
arguments.

![Figure 1](image)

We will now formally define the arguments and the attack relations
that we can derive from the constructed semantic tableaux.

**Definition 5** Let \(\Sigma\) be set of propositions and let \(\exists \alpha = \neg \exists \alpha \lor \exists \neg \alpha\)
be an assumption in the argument \(A\). Moreover, let \(T_1\) be a (weakly)
closed semantic tableaux with root \(\Gamma_1 = \{\exists \sigma \mid \sigma \in \Sigma\} \cup \exists \alpha\) and
let \(T_2\) be a (weakly) closed semantic tableaux with root \(\Gamma_2 = \{\exists \sigma \mid \sigma \in \Sigma\} \cup \exists \neg \alpha\). Finally, let \(\{\exists \alpha_1, \ldots, \exists \alpha_k\}\) be the set of assumptions
on which the weakly closed branches in the tableauxs \(T_1\) and \(T_2\) are
based.
Then the argument \( A' = (\{ \alpha_1, \ldots, \alpha_k \}, \alpha) \) is a counter-argument of the argument \( A \). We say that the argument \( A' \) attacks the argument \( A \), denoted by \( A' \rightarrow A \).

The formalization of argumentation that we have here is called assumption-based argumentation (ABA), which has been developed since the end of the 1980’s [6, 7, 11, 12, 19, 20].

Example 5 shows that an argument can be counter-argument of an argument and vice versa; e.g., arguments \( A_2 \) and \( A_1 \). This raises the question which arguments are valid. Argumentation theory and especially the argumentation framework (AF) introduced by Dung [10] provides an answer.

Arguments are viewed in an argumentation framework as atoms over which an attack relation is defined. Figure 1 shows the arguments and the attack relations between the arguments forming the argumentation framework of Example 5. The formal specification of argumentation frameworks is given by the next definition.

**Definition 6** An argumentation framework is a couple \( AF = (A, \rightarrow) \) where \( A \) is a finite set of arguments and \( \rightarrow \subseteq A \times A \) is an attack relation over the arguments.

For convenience, we extend the attack relation to sets of arguments.

**Definition 7** Let \( A \in A \) be an argument and let \( S, P \subseteq A \) be two sets of arguments. We define:

- \( S \rightarrow A \) iff for some \( B \in S \), \( B \rightarrow A \).
- \( A \rightarrow S \) iff for some \( B \in S \), \( A \rightarrow B \).
- \( S \rightarrow P \) iff for some \( B \in S \) and \( C \in P \), \( P \rightarrow C \).

Dung [10] describes different argumentation semantics for an argumentation framework in terms of sets of acceptable arguments. These semantics are based on the idea of selecting a coherent subset \( E \) of the set of arguments \( A \) of the argumentation framework \( AF = (A, \rightarrow) \). Such a set of arguments \( E \) is called an argument extension. The arguments of an argumentation support propositions that give a coherent description of what might hold in the world. Clearly, a basic requirement of an argument extension is being conflict-free; i.e., no argument in an argumentation framework attacks another argument in the argumentation framework. Besides being conflict-free, an argumentation extension should defend itself against attacking arguments by attacking the attacker.

**Definition 8** Let \( AF = (A, \rightarrow) \) be an argumentation framework and let \( S \subseteq A \) be a set of arguments.

- \( S \) is conflict-free iff \( S \rightarrow S \).
- \( S \) defends an argument \( A \in A \) iff for every argument \( B \in A \) such that \( B \rightarrow A \), \( S \rightarrow A \).

Not every conflict-free set of arguments that defends itself, is considered to be an argument extension. Several additional requirements have been formulated by Dung [10], resulting three different semantics: the stable, the preferred and the grounded semantics.

**Definition 9** Let \( AF = (A, \rightarrow) \) be an argumentation framework and let \( E \subseteq A \).

- \( E \) is a stable extension iff \( E \) is conflict-free, and for every argument \( A \in (A \setminus E), E \rightarrow A \); i.e., \( E \) defends itself against every possible attack by arguments in \( A \setminus E \).
- \( E \) is a preferred extension iff \( E \) is maximal (w.r.t. \( \subseteq \)) set of arguments that (1) is conflict-free, and (2) defends every argument \( A \in E \).

- \( E \) is a grounded extension iff \( E \) is the minimal (w.r.t. \( \subseteq \)) set of arguments that (1) is conflict-free, (2) defends every argument \( A \in E \), and (3) contains all arguments in \( A \) it defends.

We are interested in the stable semantics. If the argument for the conclusion we wish to draw, belongs to every stable extension (argument \( A_3 \) in Example 5), then this argument is justified and the conclusion holds in all conflict minimal interpretations. In Example 5 the conclusion does not hold since there is one stable extension not containing \( A_0 \). This stable extension corresponds to a conflict minimal interpretation that does not support the conclusion.

As we already mentioned in Example 5, a counter-argument \( A_0 = (\{ \alpha_0, \ldots, \alpha_k \}, \alpha_0) \) implies a set of counter-arguments of the form: \( A_i = (\{ \alpha_0, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_k \}, \alpha_i) \) with \( 0 \leq i \leq k \). This raises the question whether there can be also other arguments, which do not belong to this set, attacking an argument \( A \)? The answer is Yes, as we can easily see when we add the proposition \( r \) to the set of propositions in Example 5. After adding \( r \), we can construct an argument \( A_4 = (\{ r \}, Cr) \). The argument \( A_4 \) attacks the arguments \( A_1 \) and \( A_2 \). The resulting argumentation framework has one stable extension containing the arguments \( A_0 \) and \( A_3 \). The argument \( A_0 \) supports the conclusion \( q \).

Another question we may ask is whether it is possible that a conclusion holds, despite the fact that an argument supporting the conclusion does not hold in every stable extension? The answer is again Yes, as is illustrated by Example 6. In this example we have two arguments supporting the conclusion \( r \), namely \( A_0 \) and \( A_1 \). As can be seen in Figure 2, there are two sets of stable extensions of the argumentation framework. One extension contains the argument \( A_0 \) and the other contains the argument \( A_1 \). So, in every extension there is an argument supporting the conclusion \( r \). Hence, \( \Sigma \models E \models r \).

**Example 6** Let \( \Sigma = \{ \neg p, p \lor q, q, p \lor r, q \lor r \} \) be a set of propositions. The following two tableaux imply the two arguments \( A_0 = (\{ \top \}, r) \) and \( A_1 = (\{ \top \}, Tr) \) both supporting the conclusion \( r \):

\[
\begin{array}{c}
\top \\neg p \\
\top p \lor q \\
\top q \lor r \\
\top q \\
\top r \\
\end{array}
\]

\[
\begin{array}{c}
\top p \lor q \\
\top q \\
\top q \\
\top q \\
\top r \\
\end{array}
\]

The assumption \( \top \lor \) in argument \( A_0 \) makes it possible to determine a counter-argument \( A_2 = (\{ \top \}, \top \lor) \) using the following two tableaux:

\[
\begin{array}{c}
\top \neg p \\
\top p \lor q \\
\top q \lor r \\
\top q \\
\top r \\
\end{array}
\]

\[
\begin{array}{c}
\top p \\
\top q \\
\top q \\
\top q \\
\top r \\
\end{array}
\]
Moreover, the argument \( A_2 = (\{ C \}, C) \) implies another argument, namely: \( A_3 = (\{ C \}, C) \). The argument \( A_3 \) is a counter-argument of \( A_1 \), and the arguments \( A_2 \) and \( A_3 \) are counter-arguments of each other. No other counter-arguments can be derived in this example. Figure 2 shows the attack relations between the arguments \( A_0 \), \( A_1 \), \( A_2 \) and \( A_3 \).

\[
\begin{array}{c}
A_0 \leftarrow A_1 \\
\uparrow \\
\downarrow \\
A_2 \leftarrow A_3
\end{array}
\]

Figure 2. The attack relations between the arguments of Example 6.

Example 7 gives an illustration of the semantic interpretation of Example 6. There are two conflict-minimal interpretations. These conflict-minimal interpretations correspond with the two stable extensions. Interpretation \( I_1 \) entails \( \alpha \) because \( I_1 \) must entail \( \alpha \) or \( q \) or \( \beta \) or \( q \lor r \) and \( I_1 \) does not entail \( p \), and interpretation \( I_2 \) entails \( \alpha \) because \( I_2 \) must entail \( q \lor r \) and \( I_2 \) does not entail \( q \).

Example 7 Let \( \Sigma = \{ -p, p \lor q, -q, p \lor r, q \lor r \} \) be a set of propositions. There are two conflict-minimal interpretations containing the following interpretation functions:

- \( I_1(p) = \{ f \}, I_1(q) = \{ f, t \} \) and \( I_1(r) = \{ f \} \).
- \( I_2(p) = \{ f, t \}, I_2(q) = \{ f \} \) and \( I_2(r) = \{ t \} \).

Both interpretations entail \( r \).

## 4 Correctness and completeness proofs

In this section we investigate whether the proposed approach is correct. That is whether the proposition supported by an argument in every stable extension is entailed by every conflict-minimal interpretation. Moreover, we investigate whether the approach is complete. That is, whether a proposition entailed by every conflict-minimal interpretation is supported by an argument in every stable extension.

In the correctness and completeness theorem given below, we use the notion of “a complete set of arguments \( A \) relevant to \( \varphi \).” This set of arguments \( A \) consists of all arguments \( A \) supporting \( \varphi \), all possible counter-arguments, all possible counter arguments of the counter-arguments, etc.

**Definition 10** A complete set of arguments relevant to \( \varphi \) satisfies the following requirements:

- \( \{ A \mid A \text{ supports } \varphi \} \subseteq A \).
- If \( A \in A \) and \( B \) is a counter-argument of \( A \) that we can derive, then \( B \in A \) and \( (B, A) \in \rightarrow \).
- Nothing else belongs to \( A \).

**Theorem 1 (correctness and completeness)** Let \( \Sigma \) be a set of propositions and let \( \varphi \) be a proposition. Moreover, let \( A \) be a complete set of arguments relevant to \( \varphi \), let \( \rightarrow \subseteq A \times A \) be the attack relation determined by \( A \), and let \( (A, \rightarrow) \) be the argumentation framework. Finally, let \( \varepsilon_1, \ldots, \varepsilon_k \) be all stable extensions of the argumentation framework \( (A, \rightarrow) \).

\( \Sigma \) entails the proposition \( \varphi \) using the conflict-minimal three-valued semantics; i.e., \( \Sigma \models \varphi \), if \( \varphi \) is supported by an argument in every stable extension \( \varepsilon_i \) of \( (A, \rightarrow) \).

To prove Theorem 1, we need the following lemmas. In these lemmas we will use the following notations: We will use \( I \models T \) to denote that \( t \in I(\alpha) \); i.e., \( I \models \alpha \) and \( I \models L \alpha \) to denote that \( f \in I(\alpha) \). Moreover, we will use \( \Sigma \models T \) and \( \Sigma \models F \) to denote that \( T \alpha \) and \( F \alpha \), respectively, hold in all three-valued interpretations of \( \Sigma \).

The first lemma proves the correctness of the arguments in \( A \).

**Lemma 1 (correctness of arguments)** Let \( \Sigma \) be a set of propositions and let \( \varphi \) be a proposition. Moreover, let \( L \) be either the label \( T \) or \( F \).

If a semantic tableaux with root \( \Gamma = \{ \{ T \sigma \mid \sigma \in \Sigma \} \cup \{ L \} \} \) is weakly closed, and if \( \{ \{ C_1 \}, \ldots, \{ C_k \} \} \) is the set of weak closure assumptions implied by all the weakly closed leaves, then

\[
\{ C_1, \ldots, C_k \} \cup \{ T \sigma \mid \sigma \in \Sigma \} \models L \varphi
\]

The proof is based on the construction of a weakly closed tableaux to which we subsequently add the assumptions.

The next lemma proves the completeness of the arguments in \( A \).

**Lemma 2 (completeness of arguments)** Let \( \Sigma \) be a set of propositions and let \( \varphi \) be a proposition. Moreover, let \( L \) be either the label \( T \) or \( F \).

If \( \{ C_1, \ldots, C_k \} \) is a set of atomic assumptions with \( \alpha_i \in P \), and if

\[
\{ C_1, \ldots, C_k \} \cup \{ T \sigma \mid \sigma \in \Sigma \} \models L \varphi
\]

then there is a semantic tableaux with root \( \Gamma = \{ T \sigma \mid \sigma \in \Sigma \} \cup \{ L \} \), and the tableaux is weakly closed.

The proof is based on the construction of a closed semantic tableaux from which we subsequently remove the assumptions. The result is a weakly closed tableaux.

The following lemma proves that for every conflict \( C \varphi \) entailed by a conflict-minimal interpretation, we can find an argument supporting \( C \varphi \) of which the assumptions are entailed by the conflict-minimal interpretation.

**Lemma 3** Let \( \Sigma \) be a set of propositions and let \( I \) be a conflict-minimal interpretation of \( \Sigma \). Moreover, let \( \varphi \) be a proposition.

If \( I \models C \varphi \) holds, then there is an argument \( A = (\{ C_1, \ldots, C_k \}, C \varphi) \) supporting \( C \varphi \) and for every assumption \( C \alpha \), \( I \models C \alpha \) holds.

We prove that \( I \) is not a conflict-minimal interpretation if a tableaux for \( \{ T \sigma \mid \sigma \in \Sigma \} \cup \{ C \bracket{t, f} \} \cup \{ C \varphi \} \) is satisfiable. We remove the unused statements \( C \varphi \) from the tableaux and replace the remaining statements \( C \bracket{t, f} \) by weak closure assumptions.

For the next lemma we need the following definition of a set of assumptions that is allowed by an extension.

**Definition 11** Let \( \Omega \) be the set of all assumptions \( C \alpha \) in the arguments \( A \). For any extension \( E \subseteq A \),

\[
\Omega(E) = \{ C \alpha \in \Omega \mid \text{no argument } A \in E \text{ supports } C \alpha \}
\]

is the set of assumptions allowed by the extension \( E \).

The last lemma proves that for every conflict-minimal interpretation there is a corresponding stable extension.

**Lemma 4** Let \( \Sigma \) be a set of propositions and let \( \varphi \) be a proposition. Moreover, let \( A \) be the complete set of arguments relevant to \( \varphi \), let \( \rightarrow \subseteq A \times A \) be the attack relation determined by \( A \), and let \( (A, \rightarrow) \) be the argumentation framework.

For every conflict-minimal interpretation \( I \) of \( \Sigma \), there is a stable extension \( E \) of \( (A, \rightarrow) \) such that \( I \models \Omega(E) \).
The proof shows that, given a conflict-minimal interpretation $I$, the arguments in $\mathcal{A}$ of which the assumptions are entailed by $I$, form a conflict-free set of arguments. Moreover, Lemma 3 implies that any argument of which the assumption are not entailed by $I$, is attacked by an argument of which the assumptions are entailed by $I$. So, the arguments in $\mathcal{A}$ of which the assumptions are entailed by $I$, form a stable extension.

Using the above lemmas, we can prove the theorem.

**Proof of Theorem 1**

$(\Rightarrow)$ Let $\Sigma \models_{\leq_c} \varphi$.

Suppose that there is a stable extension $E_i$ that does not contain an argument for $\varphi$. Then according to Lemma 2, $\{T\sigma \mid \sigma \in \Sigma \} \cup \Omega(E_i) \not\models T\varphi$. So, there exists an interpretation $I$ such that $I \models \{T\sigma \mid \sigma \in \Sigma \} \cup \Omega(E_i)$ but $I \not\models T\varphi$. There must also exists a conflict-minimal interpretation $I'$ of $\Sigma$ and $I' \leq_c I$. Since the assumptions $C_p \in \Omega(E_i)$ all state that there is no conflict concerning the proposition $p$, $I' \models \Omega(E_i)$ must hold. So, $I'$ is a conflict-minimal interpretation of $\Sigma$ and $I' \models \Omega(E_i)$ but $I' \not\models T\varphi$. This implies $\Sigma \not\models_{\leq_c} \varphi$. Contradiction.

Hence, every stable extension $E_i$ contains an argument for $\varphi$.

$(\Leftarrow)$ Let $\Sigma \not\models_{\leq_c} \varphi$. Suppose that $\Sigma \not\models_{\leq_c} \varphi$. Then there is a conflict-minimal interpretation $I$ of $\Sigma$ and $I \not\models \varphi$. Since $I$ is a conflict-minimal interpretation of $\Sigma$, according to Lemma 4, there is a stable extension $E_i$ and $I \models \Omega(E_i)$. Since $E_i$ contains an argument $A$ supporting $\varphi$, the assumptions of $A$ must be a subset of $\Omega(E_i)$, and therefore $I$ satisfies these assumptions. Then, according to Lemma 1, $I \models \varphi$. Contradiction.

Hence, $\Sigma \models_{\leq_c} \varphi$. $\square$

## 5 Related Works

Paraconsistent logics have a long history starting with Aristotle. From the beginning of the twentieth century, paraconsistent logics were developed by Orlov (1929), Asenjo [1], da Costa [9] and others. For a survey of several paraconsistent logics, see for instance [15].

Signed logics [3] are closely related to paraconsistent logics. They are based on using separate representations of positive and negative instances of atomic propositions. Default assumptions are used to unify the negation of the positive instances of atomic propositions with the corresponding negative instances of the atomic propositions, thereby minimizing the number of inconsistencies.

Other approaches that been proposed in order to deal with inconsistent information are based on selecting one-of, all, or the preferred maximal consistent subsets of the set of available information [8, 14, 16, 19, 20]. Those approaches view propositions as independent, but possibly incorrect sources of information.

The semantic tableaux method was first introduced by Beth [4], and have subsequently been developed for many logics. For an overview see [13]. Bloesch [5] developed a semantic tableaux method for the paraconsistent logics LP and Belnap’s 4-valued logic.

## 6 Conclusions

This paper investigated the relation between an assumption-based argumentation system and the paraconsistent logic $LPM$. The assumption-based argumentation system enables the identification of conclusions supported by the paraconsistent logic $LPM$. The arguments of the assumption-based argumentation system are determined using a semantic tableaux method for the paraconsistent logic $LP$ (Logic of Paradox). Conclusions entailed under the $LPM$-semantics, that is, the conflict minimal interpretations of the LP-semantics, correspond to conclusions supported by all stable extensions of the resulting argumentation system.

**REFERENCES**