An argumentation system for reasoning with LPm

Wenzhao Qiao and **Nico Roos** [1]

Abstract. Inconsistent knowledge-bases can entail useful conclusions when using the three-valued semantics of the paraconsistent logic LP. However, the set of conclusions entailed by a consistent knowledge-base under the three-valued semantics is smaller than set of conclusions entailed by the knowledge-base under a two-valued semantics. Preferring conflict-minimal interpretations of the logic LP; i.e., LPm, reduces the gap between these two sets of conclusions.

Preferring conflict-minimal interpretations introduces non-monotonicity. To handle the non-monotonicity, this paper proposes an assumption-based argumentation system. Assumptions needed to close branches of a semantic tableaux form the arguments. Stable extensions of the set of derived arguments correspond to conflict minimal interpretations and conclusions entailed by all conflict-minimal interpretations are supported by arguments in all stable extensions.

1 Introduction

In an open and distributed environment such as the internet, knowledge and information originating from different sources need not be consistent. As a result, when using a standard two-valued semantics, no useful conclusions can be derived. Everything is entailed because the set of two-valued interpretations is empty. Resolving the inconsistencies is often not an option in an open and distributed environment. Therefore, methods that allow us to derive useful conclusions in the presence of inconsistencies are preferred.

One possibility to draw useful conclusions from inconsistent knowledge and information is by using a paraconsistent logic. In this paper we focus on the paraconsistent logic LP [17]. LP, which stands for Logic of Paradox, is based on a three-valued semantics. The LP-semantics assigns TRUE, FALSE or CONFLICT to each proposition. It differs from Belnap's semantics [2] in not allowing the truth-value unknown.

An important advantage of the paraconsistent logic LP is that the entailment relation is *monotonic*. A disadvantage is that consistent knowledge and information entails fewer conclusions when using the three-valued semantics, than when using the two-valued semantics. Priest [17, 18] proposed the use of conflict-minimal interpretations in LP to reduce the gap between the sets of conclusions entailed by the two semantics. The focus on conflict-minimal interpretations makes the resulting logic LPm *non-monotonic* [18].

In this paper we present an argumentation system for conclusions entailed in LPm. We start from a semantic tableaux method for LP and Belnap's logic, proposed by Bloesch [5]. The tableaux is used for deriving all conclusions entailed under the LP-semantics. If a tableaux cannot be closed, the desired conclusion may still hold in all conflict-minimal interpretations. The open tableaux enables us to identify assumptions about conflict-minimality. These assumptions are used to construct an *assumption-based argumentation system*, which supports conclusions entailed by all conflict minimal interpretations.

The remainder of the paper is organized as follows. The next section reviews the paraconsistent logics LP and LPm, and Bloesch's semantic tableaux method. Section 3 describes how Bloesch's semantic tableaux method can be used to determine arguments for conclusions supported by conflict-minimal interpretations of LPm. Subsequently, in Section 4, an outline of the correctness and completeness proof of the described approach is given. Section 5 describes some related work. The last section concludes the paper.

2 LP and LPm

In the paper we will focus on the paraconsistent logic LP and on the logic LPm, which minimizes the conflicts in the interpretations of LP. The logic LP is a three-valued logic with the truth-values TRUE, FALSE and CONFLICT. We can view the truth-values of LP as sets of truth-values of the classical Tarski semantics: $\{t\}$, $\{f\}$, and $\{t, f\}$. Hence, instead of a two-valued interpretation $I : \mathcal{P} \to \{t, f\}$ assigning t or f to atomic propositions in $\mathcal{P}$, we assign a set of classical truth-values: $I : \mathcal{P} \to (2^{\{t,f\}} \backslash \varnothing)$. The language $\mathcal{L}$ of all propositions is recursively defined starting from the set of atomic propositions $\mathcal{P}$ using the logical operators $\neg$, $\wedge$ and $\vee$. The truth-values of these propositions are determined by the extended interpretation function $I^* : \mathcal{L} \to (2^{\{t,f\}} \backslash \varnothing)$. This extended interpretation function is recursive defined by the following truth-tables:

$\neg$	
$\{t\}$	$\{f\}$
$\{f\}$	$\{t\}$
$\{t,f\}$	$\{t,f\}$

$\wedge$	$\{t\}$	$\{f\}$	$\{t,f\}$
$\{t\}$	$\{t\}$	$\{f\}$	$\{t,f\}$
$\{f\}$	$\{f\}$	$\{f\}$	$\{f\}$
$\{t,f\}$	$\{t,f\}$	$\{f\}$	$\{t,f\}$

$\vee$	$\{t\}$	$\{f\}$	$\{t,f\}$
$\{t\}$	$\{t\}$	$\{t\}$	$\{t\}$
$\{f\}$	$\{t\}$	$\{f\}$	$\{t,f\}$
$\{t,f\}$	$\{t\}$	$\{t,f\}$	$\{t,f\}$

The relation between the truth-value assignments and the three-valued entailment relation is given by: $I \models \varphi$ iff $t \in I^*(\varphi)$

Note that we get Belnap's four-valued logic if we also allow the empty set of truth-values [2]. A disadvantage of Belnap's logic compared to LP, is that tautologies need not hold because the truth-value

[1] Department of Knowledge Engineering, Maastricht University, email: {wenzhao.qiao,roos}@maastrichtuniversity.nl

of some atomic proposition is unknown. Although the result presented in the paper also apply to Belnap's logic, because of this disadvantage, we will focus on LP in the paper.

Bloesch [5] proposed a semantic tableaux method for both LP and Belnap's logic. We will use this semantic tableaux method because it enables us to handle conflict minimal interpretations. Bloesch's semantic tableaux method associates a label with every proposition in the tableaux. Possible labels are: $\mathbb{T}$ (at least true), $\mathbb{F}$ (at least false), or their complements $\overline{\mathbb{T}}$ and $\overline{\mathbb{F}}$, respectively. So, $\mathbb{T}\varphi$ corresponds to $t \in I(\varphi)$, $\overline{\mathbb{T}}\varphi$ corresponds to $t \notin I(\varphi)$, $\mathbb{F}\varphi$ corresponds to $f \in I(\varphi)$, and $\overline{\mathbb{F}}\varphi$ corresponds to $f \notin I(\varphi)$.

Although we do not need it in the semantic tableaux, we also make use of $\mathbb{C}\varphi$ and $\overline{\mathbb{C}}\varphi$, which corresponds semantically with $I(\varphi) = \{t, f\}$ and $I(\varphi) \neq \{t, f\}$, respectively. So, $\mathbb{C}\varphi$ is equivalent to: '$\mathbb{T}\varphi$ *and* $\mathbb{F}\varphi$', and $\overline{\mathbb{C}}\varphi$ is equivalent to: '$\overline{\mathbb{T}}\varphi$ *or* $\overline{\mathbb{F}}\varphi$'.

To prove that $\Sigma \models \varphi$ using Bloesch's tableaux method [5], we have to show that a tableaux with root $\Gamma = \{\mathbb{T}\sigma \mid \sigma \in \Sigma\} \cup \overline{\mathbb{T}}\varphi$ closes. The tableaux closes if every branch of the tableaux contains a node such that for some proposition α, the node one of the following three closure conditions:

- $\mathbb{T}\alpha$ and $\overline{\mathbb{T}}\alpha$,
- $\mathbb{F}\alpha$ and $\overline{\mathbb{F}}\alpha$,
- $\overline{\mathbb{T}}\alpha$ and $\overline{\mathbb{F}}\alpha$.

Note that in Belnap's logic [2], we do not have the last closure condition.

The rules of Bloesch's semantic tableaux method are given below.

$$
\begin{array}{cccc}
\dfrac{\mathbb{T}\neg p}{\mathbb{F}p} & \dfrac{\overline{\mathbb{T}}\neg p}{\overline{\mathbb{F}}p} & \dfrac{\mathbb{F}\neg p}{\mathbb{T}p} & \dfrac{\overline{\mathbb{F}}\neg p}{\overline{\mathbb{F}}p} \\
\dfrac{\mathbb{T}p \wedge q}{\mathbb{T}p, \mathbb{T}q} & \dfrac{\overline{\mathbb{T}}p \wedge q}{\overline{\mathbb{T}}p \mid \overline{\mathbb{T}}q} & \dfrac{\mathbb{F}p \wedge q}{\mathbb{F}p \mid \mathbb{F}q} & \dfrac{\overline{\mathbb{F}}p \wedge q}{\overline{\mathbb{F}}p, \overline{\mathbb{F}}q} \\
\dfrac{\mathbb{T}p \vee q}{\mathbb{T}p \mid \mathbb{T}q} & \dfrac{\overline{\mathbb{T}}p \vee q}{\overline{\mathbb{T}}p, \overline{\mathbb{T}}q} & \dfrac{\mathbb{F}p \vee q}{\mathbb{F}p, \mathbb{F}q} & \dfrac{\overline{\mathbb{F}}p \vee q}{\overline{\mathbb{F}}p \mid \overline{\mathbb{F}}q}
\end{array}
$$

A price that we pay for using the paraconsistent logic LP in order to handle inconsistencies is a reduction in the set of entailed conclusions, even if the knowledge and information is consistent.

Example 1 *The set of propositions $\Sigma = \{\neg p, p \vee q\}$ does not entail q because there exists an interpretation I for Σ such that $I(p) = \{t, f\}$ and $I(q) = \{f\}$.*

More useful conclusions can be derived if we prefer conflict-minimal interpretations [17, 18]. First, we define a conflict ordering on interpretations.

Definition 1 *Let $\mathcal{P}$ be a set of atomic propositions, and let I_1 and I_2 be two three-valued interpretations.*

The interpretation I_1 contains fewer conflicts than the interpretation I_2, denoted by $I_1 <_c I_2$, iff:

$$\{p \in \mathcal{P} \mid I_1(p) = \{t, f\}\} \subset \{p \in \mathcal{P} \mid I_2(p) = \{t, f\}\}$$

The following example gives an illustration of a conflict ordering for set of propositions of Example 1.

Example 2 *Let $\Sigma = \{\neg p, p \vee q\}$ be a set of propositions and let I_1, I_2, I_3, I_4, I_5 and I_6 be six interpretations such that:*

- $I_1(p) = \{f\}$, $I_1(q) = \{t\}$, $I_2(p) = \{f\}$, $I_2(q) = \{t, f\}$.
- $I_3(p) = \{t, f\}$, $I_3(q) = \{t\}$, $I_4(p) = \{t, f\}$, $I_4(q) = \{f\}$,
- $I_5(p) = \{t, f\}$, $I_5(q) = \{t, f\}$.

Then $I_1 <_c I_2$, $I_1 <_c I_3$, $I_1 <_c I_4$, $I_1 <_c I_5$, $I_2 <_c I_5$, $I_3 <_c I_5$, and $I_4 <_c I_5$.

Using the conflict ordering, we define the conflict minimal interpretations.

Definition 2 *Let I_1 be a three-valued interpretation and let Σ be a set of propositions.*

I_1 is a conflict minimal interpretation of Σ, denoted by $I_1 \models_{<_c} \Sigma$, iff $I_1 \models \Sigma$ and for no interpretation I_2 such that $I_2 <_c I_1$, $I_2 \models \Sigma$ holds.

In Example 2, I_1 is the only conflict-minimal interpretation.

The conflict-minimal entailment of a proposition by a set of propositions can now be defined.

Definition 3 *Let $\Sigma \subseteq \mathcal{L}$ be a set of propositions and let $\varphi \in \mathcal{L}$ be a proposition.*

Σ entails conflict-minimally the proposition φ, denoted by $\Sigma \models_{<_c} \varphi$, iff for every interpretation I, if $I \models_{<_c} \Sigma$, then $I \models \varphi$.

The conflict-minimal interpretations in Example 2 entail the conclusion q.

3 Arguments for conclusions supported by conflict minimal interpretations

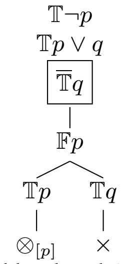
The conflict-minimal interpretations of a knowledge base entail more useful conclusions than the three-valued interpretations of the knowledge base. Unfortunately, focusing on conclusions supported by conflict-minimal interpretations makes the reasoning process *non-monotonic*. Adding the proposition $\neg q$ to the set of propositions in Example 2 eliminates interpretations I_1 and I_3, which includes the only conflict-minimal interpretation I_1. The interpretations I_2 and I_4 are the new conflict-minimal interpretations. Unlike the original conflict-minimal interpretation I_1, the new conflict-minimal interpretations I_2 and I_4 do not entail q.

Deriving conclusions supported by the conflict-minimal interpretations is problematic because of the non-monotonicity. The modern way to deal with non-monotonicity is by giving an argument supporting a conclusion and subsequently verifying whether there are no counter-arguments [10]. Here we will follow this argumentation-based approach.

We propose an approach for deriving arguments that uses the semantic tableaux method for our paraconsistent logic as a starting point. The approach is based on the observation that an interpretation satisfying the root of a semantic tableaux will also satisfy one of the leafs. Now suppose that the only leafs of a tableaux that are not closed; i.e., leafs in which we do not have "$\mathbb{T}\alpha$ *and* $\overline{\mathbb{T}}\alpha$" or "$\mathbb{F}\alpha$ *and* $\overline{\mathbb{F}}\alpha$" or "$\overline{\mathbb{T}}\alpha$ *and* $\overline{\mathbb{F}}\alpha$", are leafs in which "$\mathbb{T}\alpha$ *and* $\mathbb{F}\alpha$" holds for some propositions α. So, in every open branch of the tableaux, $\mathbb{C}\alpha$ holds for some proposition α. If we can assume that there are no conflicts w.r.t. each proposition α in the conflict-minimal interpretations, then we can also close the open branches. The set of assumptions $\overline{\mathbb{C}}\alpha$, equivalent to "$\overline{\mathbb{T}}\alpha$ *or* $\overline{\mathbb{F}}\alpha$", that we need to close the open branches, will be used as the argument for the conclusion supported by the semantic tableaux.

A branch that can be closed *assuming* that the conflict-minimal interpretations contain **no** conflict with respect to a proposition α; i.e., assuming $\overline{\mathbb{C}}\alpha$, will be called a *weakly closed* branch. We will call a tableaux *weakly closed* if some branches are weakly closed and all other branches are closed. If we can (weakly) close a tableaux for $\Gamma = \{\mathbb{T}\sigma \mid \sigma \in \Sigma\} \cup \overline{\mathbb{T}}\varphi$, we consider the set of the assumptions $\overline{\mathbb{C}}\alpha$ needed to weakly close the tableaux, to be the argument supporting $\Sigma \models_{\leq_c} \varphi$. Example 3 gives an illustration.

Example 3 *Let $\Sigma = \{\neg p, p \vee q\}$ be a set of propositions. To verify whether q holds, we may construct the following tableaux:*



Only the left branch is weakly closed *in this tableau. We assume that the proposition p will* not *be assigned* CONFLICT *in any conflict-minimal interpretation. That is, we assume that $\overline{\mathbb{C}}p$ holds.*

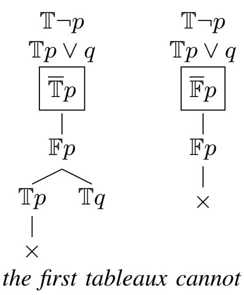
In the following definition of an argument, we consider arguments for $\mathbb{T}\varphi$ and $\mathbb{F}\varphi$.

Definition 4 *Let Σ be set of propositions and let φ a proposition. Moreover, let $\mathcal{T}$ be a (weakly) closed semantic tableaux with root $\Gamma = \{\mathbb{T}\sigma \mid \sigma \in \Sigma\} \cup \overline{\mathbb{L}}\varphi$ and $\mathbb{L} \in \{\mathbb{T}, \mathbb{F}\}$. Finally, let $\{\overline{\mathbb{C}}\alpha_1, \ldots, \overline{\mathbb{C}}\alpha_k\}$ be the set of assumptions on which the closures of weakly closed branches are based.*

Then $A = (\{\overline{\mathbb{C}}\alpha_1, \ldots, \overline{\mathbb{C}}\alpha_k\}, \mathbb{L}\varphi)$ is an argument for $\mathbb{L}\varphi$.

The next step is to verify whether the assumptions: $\overline{\mathbb{C}}\alpha$ are valid. If one of the assumptions does not hold, we have a *counter-argument* for our argument supporting $\Sigma \models_{\leq_c} \varphi$. To verify the correctness of an assumption, we add the assumption to Σ. Since an assumption $\overline{\mathbb{C}}\alpha$ is equivalent to: "$\overline{\mathbb{T}}\alpha$ *or* $\overline{\mathbb{F}}\alpha$", we can consider $\overline{\mathbb{T}}\alpha$ and $\overline{\mathbb{F}}\alpha$ separately. Example 4 gives an illustration for the assumption $\overline{\mathbb{C}}p$ used in Example 3.

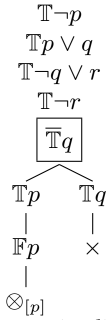
Example 4 *Let $\Sigma = \{\neg p, p \vee q\}$ be a set of propositions. To verify whether the assumption $\overline{\mathbb{C}}p$ holds in every conflict minimal interpretation, we may construct a tableaux assuming $\overline{\mathbb{T}}p$ and a tableaux assuming $\overline{\mathbb{F}}p$:*



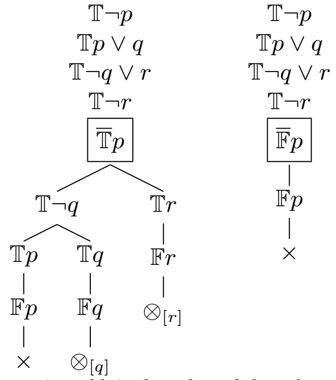
The right branch of the first tableaux cannot be closed. Therefore, the assumption $\overline{\mathbb{T}}p$ is valid, implying that the assumption $\overline{\mathbb{C}}p$ is also valid. Hence, there exists no *counter-argument.*

Since the validity of assumptions must be verified with respect to conflict-minimal interpretations, assumptions may also be used in the counter-arguments. This implies that we may have to verify whether there exists a counter-argument for a counter-argument. Example 5 gives an illustration.

Example 5 *Let $\Sigma = \{\neg p, p \vee q, \neg q \vee r, \neg r\}$ be a set of propositions. To verify whether q holds, we may construct the following tableaux:*



This weakly closed tableaux implies the argument $A_0 = (\{\overline{\mathbb{C}}p\}, \mathbb{T}q)$. Next we have to verify whether there exists a counter-argument for A_0. To verify the existence of a counter-argument, we can construct the following two tableaux:



Both tableaux are (weakly)-closed, and therefore form the counter-argument $A_1 = (\{\overline{\mathbb{C}}q, \overline{\mathbb{C}}r\}, \mathbb{C}p)$. We say that the argument A_1 attacks *the argument A_0 because the former is a counter-argument of the latter.*

The two tableaux forming the counter-argument A_1 are closed under the assumptions: $\overline{\mathbb{C}}q$ and $\overline{\mathbb{C}}r$. So, A_1 is a valid argument if there exists no valid counter-argument for $\mathbb{C}q$, and no counter-argument for $\mathbb{C}r$.

Note that the argument A_1 implies that the set of assumptions $\{\overline{\mathbb{C}}p, \overline{\mathbb{C}}r, \overline{\mathbb{C}}q\}$ together with $\{\mathbb{T}\sigma \mid \sigma \in \Sigma\}$ is not satisfiable. Therefore, A_1 implies two other arguments, namely: $A_2 = (\{\overline{\mathbb{C}}p, \overline{\mathbb{C}}r\}, \mathbb{C}q)$ and $A_3 = (\{\overline{\mathbb{C}}p, \overline{\mathbb{C}}q\}, \mathbb{C}r)$. Clearly, the argument A_1 is a counter-argument of A_2 and A_3, A_2 is a counter-argument of A_1 and A_3, and A_3 is a counter-arguments of A_1 and A_3. No other counter-arguments can be identified. Figure 1 shows arguments. The arrows denote which arguments is a counter-argument of another arguments.

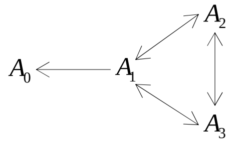


Figure 1. The attack relations between the arguments of Example 5.

We will now formally define the arguments and the attack relations that we can derive from the constructed semantic tableaux.

Definition 5 *Let Σ be set of propositions and let $\overline{\mathbb{C}}\alpha =$ "$\overline{\mathbb{T}}\alpha$ or $\overline{\mathbb{F}}\alpha$" be an assumption in the argument A. Moreover, let $\mathcal{T}_1$ be a (weakly) closed semantic tableaux with root $\Gamma_1 = \{\mathbb{T}\sigma \mid \sigma \in \Sigma\} \cup \overline{\mathbb{T}}\alpha$ and let $\mathcal{T}_2$ be a (weakly) closed semantic tableaux with root $\Gamma_2 = \{\mathbb{T}\sigma \mid \sigma \in \Sigma\} \cup \overline{\mathbb{F}}\alpha$. Finally, let $\{\overline{\mathbb{C}}\alpha_1, \ldots, \overline{\mathbb{C}}\alpha_k\}$ be the set of assumptions on which the weakly closed branches in the tableauxs $\mathcal{T}_1$ and $\mathcal{T}_2$ are based.*

Then the argument $A' = (\{\overline{\mathbb{C}}\alpha_1, \ldots, \overline{\mathbb{C}}\alpha_k\}, \mathbb{C}\alpha)$ is a counter-argument of the argument A. We say that the argument A' attacks *the argument A, denoted by $A' \longrightarrow A$.*

The form of argumentation that we have here is called assumption-based argumentation (ABA), which has been developed since the end of the 1980's [6, 7, 11, 12, 19, 20].

Example 5 shows that an argument can be counter-argument of an argument and vice versa; e.g., arguments A_2 and A_3. This raises the question which arguments are valid. Argumentation theory and especially the argumentation framework (AF) introduced by Dung [10] provides an answer.

Arguments are viewed in an argumentation framework as atoms over which an attack relation is defined. Figure 1 shows the arguments and the attack relations between the arguments forming the argumentation framework of Example 5. The formal specification of argumentation frameworks is given by the next definition.

Definition 6 *An argumentation framework is a couple $AF = (\mathcal{A}, \longrightarrow)$ where $\mathcal{A}$ is a finite set of arguments and $\longrightarrow \subseteq \mathcal{A} \times \mathcal{A}$ is an attack relation over the arguments.*

For convenience, we extend the attack relation to sets of arguments.

Definition 7 *Let $A \in \mathcal{A}$ be an argument and let $\mathcal{S}, \mathcal{P} \subseteq \mathcal{A}$ be two sets of arguments. We define:*

- *$\mathcal{S} \longrightarrow A$ iff for some $B \in \mathcal{S}$, $B \longrightarrow A$.*
- *$A \longrightarrow \mathcal{S}$ iff for some $B \in \mathcal{S}$, $A \longrightarrow B$.*
- *$\mathcal{S} \longrightarrow \mathcal{P}$ iff for some $B \in \mathcal{S}$ and $C \in \mathcal{P}$, $B \longrightarrow C$.*

Dung [10] describes different argumentation semantics for an argumentation framework in terms of sets of acceptable arguments. These semantics are based on the idea of selecting a coherent subset $\mathcal{E}$ of the set of arguments $\mathcal{A}$ of the argumentation framework $AF = (\mathcal{A}, \longrightarrow)$. Such a set of arguments $\mathcal{E}$ is called an *argument extension*. The arguments of an argument extension support propositions that give a coherent description of what might hold in the world. Clearly, a basic requirement of an argument extension is being *conflict-free*; i.e., no argument in an argument extension attacks another argument in the argument extension. Besides being conflict-free, an argument extension should defend itself against attacking arguments by attacking the attacker.

Definition 8 *Let $AF = (\mathcal{A}, \longrightarrow)$ be an argumentation framework and let $\mathcal{S} \subseteq \mathcal{A}$ be a set of arguments.*

- *$\mathcal{S}$ is* conflict-free *iff $\mathcal{S} \not\longrightarrow \mathcal{S}$.*
- *$\mathcal{S}$ defends an argument $A \in \mathcal{A}$ iff for every argument $B \in \mathcal{A}$ such that $B \longrightarrow A$, $\mathcal{S} \longrightarrow A$.*

Not every conflict-free set of arguments that defends itself, is considered to be an argument extension. Several additional requirements have been formulated by Dung [10], resulting three different semantics: the *stable*, the *preferred* and the *grounded* semantics.

Definition 9 *Let $AF = (\mathcal{A}, \longrightarrow)$ be an argumentation framework and let $\mathcal{E} \subseteq \mathcal{A}$.*

- *$\mathcal{E}$ is a* stable extension *iff $\mathcal{E}$ is conflict-free, and for every argument $A \in (\mathcal{A} \backslash \mathcal{E})$, $\mathcal{E} \longrightarrow A$; i.e., $\mathcal{E}$ defends itself against every possible attack by arguments in $\mathcal{A} \backslash \mathcal{E}$.*
- *$\mathcal{E}$ is a* preferred extension *iff $\mathcal{E}$ is maximal (w.r.t. $\subseteq$) set of arguments that (1) is conflict-free, and (2) defends every argument $A \in \mathcal{E}$.*
- *$\mathcal{E}$ is a* grounded extension *iff $\mathcal{E}$ is the minimal (w.r.t. $\subseteq$) set of arguments that (1) is conflict-free, (2) defends every argument $A \in \mathcal{E}$, and (3) contains all arguments in $\mathcal{A}$ it defends.*

We are interested in the stable semantics. If the argument for the conclusion we wish to draw, belongs to every stable extension (argument A_0 in Example 5), then this argument is justified and the conclusion holds in all conflict minimal interpretations. In Example 5 the conclusion does not hold since there is one stable extension not containing A_0. This stable extension corresponds to a conflict minimal interpretation that does not support the conclusion.

As we already mentioned in Example 5, a counter-argument $A_0 = (\{\overline{\mathbb{C}}\alpha_1, \ldots, \overline{\mathbb{C}}\alpha_k\}, \mathbb{C}\alpha_0)$ implies a set of counter-arguments of the form: $A_i = (\{\overline{\mathbb{C}}\alpha_0, \ldots, \overline{\mathbb{C}}\alpha_{i-1}, \overline{\mathbb{C}}\alpha_{i+1}, \ldots, \overline{\mathbb{C}}\alpha_k\}, \mathbb{C}\alpha_i)$ with $0 \leq i \leq k$. This raises the question whether there can also be other arguments, which do not belong to this set, attacking an argument A_i? The answer is Yes, as we can easily see when we add the proposition r to the set of propositions in Example 5. After adding r, we can construct an argument $A_4 = (\varnothing, \mathbb{C}r)$. The argument A_4 attacks the arguments A_1 and A_2. The resulting argumentation framework has one stable extension containing the arguments A_0 and A_3. The argument A_0 supports the conclusion q.

Another question we may ask is whether it is possible that a conclusion holds, despite the fact that an argument supporting the conclusion does not hold in every stable extension? The answer is again Yes, as is illustrated by Example 6. In this example we have two arguments supporting the conclusion r, namely A_0 and A_1. As can be seen in Figure 2, there are two stable extensions of the argumentation framework. One extension contains the argument A_0 and the other contains the argument A_1. So, in every extension there is an argument supporting the conclusion r. Hence, $\Sigma \models_{\leq_c} r$.

Example 6 *Let $\Sigma = \{\neg p, p \vee q, \neg q, p \vee r, q \vee r\}$ be a set of propositions. The following two tableaux imply the two arguments $A_0 = (\{\overline{\mathbb{C}}p\}, \mathbb{T}r)$ and $A_1 = (\{\overline{\mathbb{C}}q\}, \mathbb{T}r)$ both supporting the conclusion r:*

$$
\begin{array}{c}
\mathbb{T}\neg p \\
\mathbb{T}p \vee q \\
\mathbb{T}\neg q \\
\mathbb{T}p \vee r \\
\mathbb{T}q \vee r \\
\boxed{\overline{\mathbb{T}}r} \\
| \\
\mathbb{F}p \\
\begin{array}{cc} \mathbb{T}p & \mathbb{T}r \\ | & | \\ \otimes_{[p]} & \times \end{array}
\end{array}
\qquad
\begin{array}{c}
\mathbb{T}\neg p \\
\mathbb{T}p \vee q \\
\mathbb{T}\neg q \\
\mathbb{T}p \vee r \\
\mathbb{T}q \vee r \\
\boxed{\overline{\mathbb{T}}r} \\
| \\
\mathbb{F}q \\
\begin{array}{cc} \mathbb{T}q & \mathbb{T}r \\ | & | \\ \otimes_{[q]} & \times \end{array}
\end{array}
$$

The assumption $\overline{\mathbb{C}}p$ in argument A_0 makes it possible to determine a counter-argument $A_2 = (\{\overline{\mathbb{C}}q\}, \mathbb{C}p)$ using of the following two tableaux:

$$
\begin{array}{c}
\mathbb{T}\neg p \\
\mathbb{T}p \vee q \\
\mathbb{T}\neg q \\
\mathbb{T}p \vee r \\
\mathbb{T}q \vee r \\
\boxed{\overline{\mathbb{T}}p} \\
| \\
\mathbb{F}q \\
\begin{array}{cc} \mathbb{T}p & \mathbb{T}q \\ | & | \\ \times & \otimes_{[q]} \end{array}
\end{array}
\qquad
\begin{array}{c}
\mathbb{T}\neg p \\
\mathbb{T}p \vee q \\
\mathbb{T}\neg q \\
\mathbb{T}p \vee r \\
\mathbb{T}q \vee r \\
\boxed{\overline{\mathbb{F}}p} \\
| \\
\mathbb{F}p \\
| \\
\times
\end{array}
$$

Moreover, the argument $A_2 = (\{\overline{\mathbb{C}}q\}, \mathbb{C}p)$ implies another argument, namely: $A_3 = (\{\overline{\mathbb{C}}p\}, \mathbb{C}q)$. The argument A_3 is a counter-argument of A_1, and the arguments A_2 and A_3 are counterarguments of each other. No other counter-arguments can be derived in this example. Figure 2 shows the attack relations between the arguments A_0, A_1, A_2 and A_3.

$$\begin{tikzcd} A_0 & A_2 \arrow[l] \arrow[d, leftrightarrow] \\ A_1 & A_3 \arrow[l] \end{tikzcd}$$

Figure 2. The attack relations between the arguments of Example 6.

Example 7 gives an illustration of the semantic interpretation of Example 6. There are two conflict-minimal interpretations. These conflict-minimal interpretations correspond with the two *stable extensions*. Interpretation I_1 entails r because I_1 must entail $p \vee r$ and I_1 does not entail p, and interpretation I_2 entails r because I_2 must entail $q \vee r$ and I_2 does not entail q.

Example 7 *Let $\Sigma = \{\neg p, p \vee q, \neg q, p \vee r, q \vee r\}$ be a set of propositions. There are two conflict-minimal interpretations containing the following interpretation functions:*

- *$I_1(p) = \{f\}$, $I_1(q) = \{t, f\}$ and $I_1(r) = \{t\}$.*
- *$I_2(p) = \{t, f\}$, $I_2(q) = \{f\}$ and $I_2(r) = \{t\}$.*

Both interpretations entail r.

4 Correctness and completeness proofs

In this section we investigate whether the proposed approach is correct. That is whether the proposition supported by an argument in every stable extension is entailed by every conflict-minimal interpretation. Moreover, we investigate whether the approach is complete. That is, whether a proposition entailed by every conflict-minimal interpretation is supported by an argument in every stable extension.

In the correctness and completeness theorem given below, we use the notion of "a complete set of arguments $\mathcal{A}$ relevant to φ". This set of arguments $\mathcal{A}$ consists of all arguments A supporting φ, all possible counter-arguments, all possible counter arguments of the counter-arguments, etc.

Definition 10 *$\mathcal{A}$ complete set of arguments relevant to φ satisfies the following requirements:*

- *$\{A \mid A$ supports $\varphi\} \subseteq \mathcal{A}$.*
- *If $A \in \mathcal{A}$ and B is a counter-argument of A that we can derive, then $B \in \mathcal{A}$ and $(B, A) \in \longrightarrow$.*
- *Nothing else belongs to $\mathcal{A}$.*

Theorem 1 (correctness and completeness) *Let Σ be a set of propositions and let φ be a proposition. Moreover, let $\mathcal{A}$ be a complete set of arguments relevant to φ, let $\longrightarrow \subseteq \mathcal{A} \times \mathcal{A}$ be the attack relation determined by $\mathcal{A}$, and let $(\mathcal{A}, \longrightarrow)$ be the argumentation framework. Finally, let $\mathcal{E}_1, \ldots, \mathcal{E}_k$ be all stable extensions of the argumentation framework $(\mathcal{A}, \longrightarrow)$.*

Σ entails the proposition φ using the conflict-minimal three-valued semantics; i.e., $\Sigma \models_{\leq_c} \varphi$, iff φ is supported by an argument in every stable extension $\mathcal{E}_i$ of $(\mathcal{A}, \longrightarrow)$.

To prove Theorem 1, we need the following lemmas. In these lemmas we will use the following notations: We will use $I \models \mathbb{T}\alpha$ to denote that $t \in I(\alpha)$; i.e., $I \models \alpha$, and $I \models \mathbb{F}\alpha$ to denote that $f \in I(\alpha)$. Moreover, we will use $\Sigma \models \mathbb{T}\alpha$ and $\Sigma \models \mathbb{F}\alpha$ to denote that $\mathbb{T}\alpha$ and $\mathbb{F}\alpha$, respectively, hold in all three-valued interpretations of Σ.

The first lemma proves the correctness of the arguments in $\mathcal{A}$.

Lemma 1 (correctness of arguments) *Let Σ be a set of propositions and let φ be a proposition. Moreover, let $\mathbb{L}$ be either the label $\mathbb{T}$ or $\mathbb{F}$.*

If a semantic tableaux with root $\Gamma = \{\mathbb{T}\sigma \mid \sigma \in \Sigma\} \cup \{\overline{\mathbb{L}}\varphi\}$ is weakly closed, and if $\{\overline{\mathbb{C}}\alpha_1, \ldots, \overline{\mathbb{C}}\alpha_k\}$ is the set of weak closure assumptions implied by all the weakly closed leafs, then

$$\{\overline{\mathbb{C}}\alpha_1, \ldots, \overline{\mathbb{C}}\alpha_k\} \cup \{\mathbb{T}\sigma \mid \sigma \in \Sigma\} \models \mathbb{L}\varphi$$

The proof is based on the construction of a weakly closed tableaux to which we subsequently add the assumptions.

The next lemma proves the completeness of the arguments in $\mathcal{A}$.

Lemma 2 (completeness of arguments) *Let Σ be a set of propositions and let φ be a proposition. Moreover, let $\mathbb{L}$ be either the label $\mathbb{T}$ or $\mathbb{F}$.*

If $\{\overline{\mathbb{C}}\alpha_1, \ldots, \overline{\mathbb{C}}\alpha_k\}$ is a set of atomic assumptions with $\alpha_i \in \mathcal{P}$, and if

$$\{\overline{\mathbb{C}}\alpha_1, \ldots, \overline{\mathbb{C}}\alpha_k\} \cup \{\mathbb{T}\sigma \mid \sigma \in \Sigma\} \models \mathbb{L}\varphi$$

then there is a semantic tableaux with root $\Gamma = \{\mathbb{T}\sigma \mid \sigma \in \Sigma\} \cup \{\overline{\mathbb{L}}\varphi\}$, and the tableaux is weakly closed.

The proof is based on the construction of a closed semantic tableaux from which we subsequently remove the assumptions. The result is a weakly closed tableaux.

The following lemma proves that for every conflict $\mathbb{C}\varphi$ entailed by a conflict-minimal interpretation, we can find an argument supporting $\mathbb{C}\varphi$ of which the assumptions are entailed by the conflict-minimal interpretation.

Lemma 3 *Let Σ be a set of propositions and let I be a conflict-minimal interpretation of Σ. Moreover, let φ be a proposition.*

If $I \models \mathbb{C}\varphi$ holds, then there is an argument $A = (\{\overline{\mathbb{C}}\alpha_1, \ldots, \overline{\mathbb{C}}\alpha_k\}, \mathbb{C}\varphi)$ supporting $\mathbb{C}\varphi$ and for every assumption $\overline{\mathbb{C}}\alpha_i$, $I \models \overline{\mathbb{C}}\alpha_i$ holds.

We prove that I is *not* a conflict-minimal interpretation if a tableaux for $\{\mathbb{T}\sigma \mid \sigma \in \Sigma\} \cup \{\overline{\mathbb{C}}p \mid p \in \mathcal{P}, I(p) \neq \{t, f\}\} \cup \{\overline{\mathbb{C}}\varphi\}$ is satisfiable. We remove the unused statements $\overline{\mathbb{C}}p$ from the tableaux and replace the remaining statements $\overline{\mathbb{C}}p$ by weak closure assumptions.

For the next lemma we need the following definition of a set of assumptions that is allowed by an extension.

Definition 11 *Let Ω be the set of all assumptions $\overline{\mathbb{C}}\alpha$ in the arguments $\mathcal{A}$. For any extension $\mathcal{E} \subseteq \mathcal{A}$,*

$$\Omega(\mathcal{E}) = \{\overline{\mathbb{C}}\alpha \in \Omega \mid \text{no argument } A \in \mathcal{E} \text{ supports } \mathbb{C}\alpha\}$$

is the set of assumptions allowed by the extension $\mathcal{E}$.

The last lemma proves that for every conflict-minimal interpretation there is a corresponding stable extension.

Lemma 4 *Let Σ be a set of propositions and let φ be a proposition. Moreover, let $\mathcal{A}$ be the complete set of arguments relevant to φ, let $\longrightarrow \subseteq \mathcal{A} \times \mathcal{A}$ be the attack relation determined by $\mathcal{A}$, and let $(\mathcal{A}, \longrightarrow)$ be the argumentation framework.*

For every conflict-minimal interpretation I of Σ, there is a stable extension $\mathcal{E}$ of $(\mathcal{A}, \longrightarrow)$ such that $I \models \Omega(\mathcal{E})$.

The proof shows that, given a conflict-minimal interpretation I, the arguments in $\mathcal{A}$ of which the assumptions are entailed by I, form a conflict-free set of arguments. Moreover, Lemma 3 implies that any argument of which the assumption are not entailed by I, is attacked by an argument of which the assumptions are entailed by I. So, the arguments in $\mathcal{A}$ of which the assumptions are entailed by I, form a stable extension.

Using the above lemmas, we can prove the theorem.

Proof of Theorem 1
($\Rightarrow$) Let $\Sigma \models_{\leq_c} \varphi$.

Suppose that there is stable extension $\mathcal{E}_i$ that does not contain an argument for φ. Then according to Lemma 2, $\{\mathbb{T}\sigma \mid \sigma \in \Sigma\} \cup \Omega(\mathcal{E}_i) \not\models \mathbb{T}\varphi$. So, there exists an interpretation I such that $I \models \{\mathbb{T}\sigma \mid \sigma \in \Sigma\} \cup \Omega(\mathcal{E}_i)$ but $I \not\models \mathbb{T}\varphi$. There must also exists a conflict-minimal interpretation I' of Σ and $I' \leq_c I$. Since the assumptions $\overline{\mathbb{C}}p \in \Omega(\mathcal{E}_i)$ all state that there is no conflict concerning the proposition p, $I' \models \Omega(\mathcal{E}_i)$ must hold. So, I' is a conflict-minimal interpretation of Σ and $I' \models \Omega(\mathcal{E}_i)$ but $I' \not\models \mathbb{T}\varphi$. This implies $\Sigma \not\models_{\leq_c} \varphi$. Contradiction.

Hence, every stable extension $\mathcal{E}_i$ contains an argument for φ.

($\Leftarrow$) Let φ be supported by an argument in every stable extension $\mathcal{E}_i$.

Suppose that $\Sigma \not\models_{\leq_c} \varphi$. Then there is a conflict-minimal interpretation I of Σ and $I \not\models \varphi$. Since I is a conflict-minimal interpretation of Σ, according to Lemma 4, there is a stable extension $\mathcal{E}_i$ and $I \models \Omega(\mathcal{E}_i)$. Since $\mathcal{E}_i$ contains an argument A supporting φ, the assumptions of A must be a subset of $\Omega(\mathcal{E}_i)$, and therefore I satisfies these assumptions. Then, according to Lemma 1, $I \models \varphi$. Contradiction.

Hence, $\Sigma \models_{\leq_c} \varphi$. □

5 Related Works

Paraconsistent logics have a long history starting with Aristotle. From the beginning of the twentieth century, paraconsistent logics were developed by Orlov (1929), Asenjo [1], da Costa [9] and others. For a survey of several paraconsistent logics, see for instance [15].

Signed logics [3] are closely related to paraconsistent logics. They are based on using separate representations of positive and negative instances of atomic propositions. Default assumptions are used to unify the negation of the positive instances of atomic propositions with the corresponding negative instances of the atomic propositions, thereby minimizing the number of inconsistencies.

Other approaches that been proposed in order to deal with inconsistent information are based on selecting one-of, all, or the preferred maximal consistent subsets of the set of available information [8, 14, 16, 19, 20]. Those approaches view propositions as independent, but possibly incorrect sources of information.

The semantic tableaux method was first introduced by Beth [4], and have subsequently been developed for many logics. For an overview see [13]. Bloesch [5] developed a semantic tableaux method for the paraconsistent logics LP and Belnap's 4-valued logic.

6 Conclusions

This paper investigated the relation between an assumption-based argumentation system and the paraconsistent logic LPm. The assumption-based argumentation system enables the identification of conclusions supported by the paraconsistent logic LPm. The arguments of the assumption-based argumentation system are determined using a semantic tableaux method for the paraconsistent logic LP (Logic of Paradox). Conclusions entailed under the LPm-semantics, that is, the conflict minimal interpretations of the LP-semantics, correspond to conclusions supported by all stable extensions of the resulting argumentation system.

REFERENCES

[1] F.G. Asenjo, 'A calculus of antinomies', *Notre Dame Journal of Formal Logic*, **7**, 103105, (1966).

[2] Nuel D. Belnap, 'A useful four-valued logic', in *Modern Uses of Multiple-Valued Logic*, eds., J. Michael Dunn and George Epstein, 8–37, Reidel, Dordrecht, (1977).

[3] Philippe Besnard and Torsten Schaub, 'Signed systems for paraconsistent reasoning', *Journal of Automated Reasoning*, **20**(1), 191–213, (1998).

[4] Evert Willem Beth, *Semantic entailment and formal derivability*, Noord-Hollandsche Uitg. Mij., 1955.

[5] Anthony Bloesch, 'A tableau style proof system for two paraconsistent logics', *Notre Dame Journal of Formal Logic*, **34**(2), 295–301, (1993).

[6] A. Bondarenko, P.M. Dung, R.A. Kowalski, and F. Toni, 'An abstract, argumentation-theoretic approach to default reasoning', *Artificial Intelligence*, **93**(1-2), 63–101, (1997).

[7] A. Bondarenko, F. Toni, and R. Kowalski, 'An assumption-based framework for nonmonotonic reasoning', in *Proc. 2nd International Workshop on Logic Programming and Non-monotonic Reasoning*. MIT Press, (1993).

[8] G. Brewka, 'Preferred subtheories: an extended logical framework for default reasoning', in *International Joined Conference on Artificial Intelligence*, pp. 1043–1048, (1989).

[9] N.C.A. da Costa, 'On the theory of inconsistent formal systems', *Notre Dame Journal of Formal Logic*, **15**, 497510, (1974).

[10] P. M. Dung, 'On the acceptability of arguments and its fundamental role in nonmonotonic reasoning, logic programming and *n*-person games', *Artificial Intelligence*, **77**, 321–357, (1995).

[11] P. M. Dung, R. Kowalski, and F Toni, 'Assumption-based argumentation', in *Argumentation in Artificial Intelligence*, eds., I. Rahwan and G. Simari, 1–20, Springer, (2009).

[12] Dorian Gaertner and Francesca Toni, 'Computing arguments and attacks in assumption-based argumentation', *IEEE Intelligent Systems*, **22**(6), 24–33, (2007).

[13] Reiner Hähnle, 'Tableaux and related methods', in *Handbook of Automated Reasoning*, eds., John Alan Robinson and Andrei Voronkov, volume 1, chapter 3, 100–178, Elsevier and MIT Press, (2001).

[14] Zhisheng Huang, Frank van Harmelen, and Annette ten Teije, 'Reasoning with inconsistent ontologies', in *IJCAI*, pp. 454–459, (2005).

[15] C. A. Middelburg, 'A survey of paraconsistent logics', *CoRR abs/1103.4324*, (2011).

[16] D. Poole, 'A logical framework for default reasoning', *Artificial Intelligence*, **36**, 27–47, (1988).

[17] G. Priest, 'Reasoning about truth', *Artificial Intelligence*, **39**(2), 231–244, (1989).

[18] Graham Priest, 'Minimally inconsistent lp', *Studia Logica*, **50**, 321–331, (1991).

[19] N. Roos, 'A preference logic for non-monotonic reasoning', Technical Report 88-94, Delft University of Technology, Faculty of Technical Mathematics and Informatics. ISSN 0922-5641, (1988).

[20] N. Roos, 'A logic for reasoning with inconsistent knowledge', *Artificial Intelligence*, **57**, 69–103, (1992).