An argumentation system for reasoning with conflict-minimal paraconsistent $\mathcal{ALC}$

Wenzhao Qiao and **Nico Roos**
Department of Knowledge Engineering, Maastricht University
Bouillonstraat 8-10, 6211 LH Maastricht, The Netherlands
{wenzhao.qiao,roos}@maastrichtuniversity.nl

Abstract

The semantic web is an open and distributed environment in which it is hard to guarantee consistency of knowledge and information. Under the standard two-valued semantics everything is entailed if knowledge and information is inconsistent. The semantics of the paraconsistent logic LP offers a solution. However, if the available knowledge and information is consistent, the set of conclusions entailed under the three-valued semantics of the paraconsistent logic LP is smaller than the set of conclusions entailed under the two-valued semantics. Preferring conflict-minimal three-valued interpretations eliminates this difference.

Preferring conflict-minimal interpretations introduces non-monotonicity. To handle the non-monotonicity, this paper proposes an assumption-based argumentation system. Assumptions needed to close branches of a semantic tableaux form the arguments. Stable extensions of the set of derived arguments correspond to conflict minimal interpretations and conclusions entailed by all conflict-minimal interpretations are supported by arguments in all stable extensions.

Introduction

In the semantic web, the description logics $\mathcal{SHOIN}(D)$ and $\mathcal{SROIQ}(D)$ are the standard for describing ontologies using the TBox, and information using the Abox. Since the semantic web is an open and distributed environment, knowledge and information originating from different sources need not be consistent. In case of inconsistencies, no useful conclusion can be derived when using a standard two-valued semantics. Everything is entailed because the set of two-valued interpretations is empty. Resolving the inconsistencies is often not an option in an open and distributed environment. Therefore, methods that allow us to derive useful conclusions in the presence of inconsistencies are preferred.

One possibility to draw useful conclusions from inconsistent knowledge and information is by focussing on conclusions supported by all maximally consistent subsets. This approach was first proposed by Rescher (29) and was subsequence worked out further by others (8; 30; 31). A simple implementation of this approach focusses on conclusions entailed by the intersection of all maximally consistent subsets. Instead of focussing on the intersection of all maximally consistent subsets, one may also consider a single consistent subset for each conclusion (25; 15). For conclusions entailed by all (preferred) maximally consistent subsets of the knowledge and information, a more sophisticated approach is needed. An argumentation system for this more general case has been described by Roos (31). Since these approaches need to identify consistent subsets of knowledge and information, they are *non-monotonic*.

A second possibility for handling inconsistent knowledge and information is by replacing the standard two-valued semantics by a three-valued semantics such as the semantics of the paraconsistent logic LP (26). An important advantage of this paraconsistent logic over the maximally consistent subset approach is that the entailment relation is *monotonic*. A disadvantage is that consistent knowledge and information entail less conclusions when using the three-valued semantics than when using the two-valued semantics. Conflict-minimal interpretations reduce the gap between the sets of conclusions entailed by the two semantics (26; 27). Priest (27) calls resulting logic: LPm. The conflict-minimal interpretations also makes LPm *non-monotonic* (27).

In this paper we present an argumentation system for conclusions entailed by conflict-minimal interpretations of the description logic $\mathcal{ALC}$ (32) when using the semantics of the paraconsistent logic LP. We focus on $\mathcal{ALC}$ instead of the more expressive logics $\mathcal{SHOIN}(D)$ and $\mathcal{SROIQ}(D)$ to keep the explanation simple. The described approach can also be applied to more expressive description logics.

The proposed approach starts from a semantic tableaux method for the paraconsistent logic LP described by Bloesch (5), which has been adapted to $\mathcal{ALC}$. The semantic tableaux is used for deriving the entailed conclusions when using the LP-semantics. If a tableaux cannot be closed, the desired conclusion may still hold in all conflict-minimal interpretations. The open tableaux enables us to identify assumptions about conflict-minimality. These assumptions are used to construct an *assumption-based argumentation system*, which supports conclusions entailed by all conflict minimal interpretations.

The remainder of the paper is organized as follows. First, we describe $\mathcal{ALC}$, a three-valued semantics for $\mathcal{ALC}$ based on the semantics of the paraconsistent logic LP, and a corresponding semantic tableaux method. Second, we describe how a semantic tableaux can be used to determine arguments for conclusions supported by conflict-minimal interpretations. Subsequently, we present the correctness and completeness proof of the described approach. Next we describe some related work. The last section summarizes the results and points out directions of future work.

Paraconsistent $\mathcal{ALC}$

The language of $\mathcal{ALC}$ We first give the standard definitions of the language of $\mathcal{ALC}$. We start with defining the set of concepts $\mathcal{C}$ given the atomic concepts $\mathbf{C}$, the role relations $\mathbf{R}$, the operators for constructing new concepts $\neg$, $\sqcap$ and $\sqcup$, and the quantifiers $\exists$ and $\forall$. Moreover, we introduce to special concepts, $\top$ and $\bot$, which denote *everything* and *nothing*, respectively.

Definition 1 *Let $\mathbf{C}$ be a set of atomic concepts and let $\mathbf{R}$ be a set of atomic roles.*

The set of concepts $\mathcal{C}$ is recursively defined as follows:

- *$\mathbf{C} \subseteq \mathcal{C}$; i.e. atomic concepts are concepts.*
- *$\top \in \mathcal{C}$ and $\bot \in \mathcal{C}$.*
- *If $C \in \mathcal{C}$ and $D \in \mathcal{C}$, then $\neg C \in \mathcal{C}$, $C \sqcap D \in \mathcal{C}$ and $C \sqcup D \in \mathcal{C}$.*
- *If $C \in \mathcal{C}$ and $R \in \mathbf{R}$, then $\exists R.C \in \mathcal{C}$ and $\forall R.C \in \mathcal{C}$.*
- *Nothing else belongs to $\mathcal{C}$.*

In the description logic $\mathcal{ALC}$, we have two operators: $\sqsubseteq$ and $=$, for describing a relation between two concepts:

Definition 2 *If $\{C, D\} \subseteq \mathcal{C}$, then we can formulate the following relations (terminological definitions):*

- *$C \sqsubseteq D$; i.e., C is subsumed by D,*
- *$C = D$; i.e., C is equal to D.*

A finite set $\mathcal{T}$ of terminological definitions is called a *TBox*.

In the description logic $\mathcal{ALC}$, we also have an operator ":", for describing that an individual from the set of individual names $\mathbf{N}$ is an instance of a concept, and that a pair of individuals is an instance of a role.

Definition 3 *Let $\{a, b\} \subseteq \mathbf{N}$ be two individuals, let $C \in \mathcal{C}$ be a concept and let $R \in \mathbf{R}$ be a role. Then assertions are defined as:*

- *$a : C$*
- *$(a, b) : R$*

A finite set $\mathcal{A}$ of assertions is called an *ABox*.

A *knowledge base* $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ is a tuple consisting of a TBox $\mathcal{T}$ and an ABox $\mathcal{A}$. In this paper we will denote elements of the TBox and ABox $\mathcal{T} \cup \mathcal{A}$ as propositions.

We define a three-valued semantics for $\mathcal{ALC}$ which is based on the semantics of the paraconsistent logic LP. We do not use the notation $I = (\Delta, \cdot^I)$ that is often used for the semantics of description logics. Instead we will use a notation that is often used for predicate logic because it is more convenient to describe projections and truth-values.

Definition 4 *A three-valued interpretation $I = \langle O, \pi \rangle$ is a couple where O is a non-empty set of objects and π is an interpretation function such that:*

- *for each atomic concept $C \in \mathbf{C}$, $\pi(C) = \langle P, N \rangle$ where $P, N \subseteq O$ are the positive and negative instances of the concept C, respectively, and where $P \cup N = O$,*
- *for each individual $i \in \mathbf{N}$ it holds that $\pi(i) \in O$, and*
- *for each atomic role $R \in \mathbf{R}$ it holds that $\pi(R) \subseteq O \times O$.*

We will use the projections $\pi(C)^+ = P$ and $\pi(C)^- = N$ to denote the positive and negative instances of a concept C, respectively.

We do not consider inconsistencies in roles since we cannot formulate inconsistent roles in $\mathcal{ALC}$. In a more expressive logic, such as $\mathcal{SROIQ}$, roles may become inconsistent, for instance because we can specify disjoint roles.

Using the three-valued interpretations $I = \langle O, \pi \rangle$, we define the interpretations of concepts in $\mathcal{C}$.

Definition 5 *The interpretation of a concept $C \in \mathcal{C}$ is defined by the extended interpretation function π^*.*

- $\pi^*(C) = \pi(C)$ *iff* $C \in \mathbf{C}$
- $\pi^*(\top) = \langle O, X \rangle$, *where* $X \subseteq O$
- $\pi^*(\bot) = \langle X, O \rangle$, *where* $X \subseteq O$
- $\pi^*(\neg C) = \langle \pi^*(C)^-, \pi^*(C)^+ \rangle$
- $\pi^*(C \sqcap D) = \langle \pi^*(C)^+ \cap \pi^*(D)^+, \pi^*(C)^- \cup \pi^*(D)^- \rangle$
- $\pi^*(C \sqcup D) = \langle \pi^*(C)^+ \cup \pi^*(D)^+, \pi^*(C)^- \cap \pi^*(D)^- \rangle$
- $\pi^*(\exists R.C) =$
  $\langle\ \{x \in O \mid \exists y \in O, (x, y) \in \pi(R) \text{ and } y \in \pi(C)^+\},$
  $\{x \in O \mid \forall y \in O, (x, y) \in \pi(R) \text{ implies } y \in \pi(C)^-\}\ \rangle$
- $\pi^*(\forall R.C) =$
  $\langle\ \{x \in O \mid \forall y \in O, (x, y) \in \pi(R) \text{ implies } y \in \pi(C)^+\},$
  $\{x \in O \mid \exists y \in O, (x, y) \in \pi(R) \text{ and } y \in \pi(C)^-\}\ \rangle$

Note that we allow inconsistencies in the concepts $\top$ and $\bot$. There may not exist a tree-valued interpretation for a knowledge-base $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ if we require that $X = \varnothing$. Consider for instance: $a : C$, $a : D$ and $C \sqcap D \sqsubseteq \bot$.

We also use the extended interpretation function π^* to define the truth values of the propositions: $C \sqsubseteq D$, $a : C$ and $(a, b) : R$. The truth values of the three-valued semantics are defined using sets of the "classical" truth values: t and f. We use three sets in the LP-semantics: $\{t\}$, $\{f\}$ and $\{t, f\}$, which correspond to TRUE, FALSE and CONFLICT.

Definition 6 *Let $\{a, b\} \subseteq \mathbf{N}$ be two individuals, let $C \in \mathcal{C}$ be a concept and let $R \in \mathbf{R}$ be a role. Then an interpretation $I = \langle O, \pi \rangle$ of propositions is defined as:*

- $t \in \pi^*(C \sqsubseteq D)$ *iff* $\pi^*(C)^+ \subseteq \pi^*(D)^+$ *and* $\pi^*(D)^- \subseteq \pi^*(C)^-$
- $f \in \pi^*(C \sqsubseteq D)$ *iff* $t \notin \pi^*(C \sqsubseteq D)$
- $t \in \pi^*(C = D)$ *iff* $\pi^*(C)^+ = \pi^*(D)^+$ *and* $\pi^*(D)^- = \pi^*(C)^-$
- $f \in \pi^*(C = D)$ *iff* $t \notin \pi^*(C = D)$
- $t \in \pi^*(a : C)$ *iff* $\pi^*(a) \in \pi^*(C)^+$
- $f \in \pi^*(a : C)$ *iff* $\pi^*(a) \in \pi^*(C)^-$
- $t \in \pi^*((a, b) : R)$ *iff* $(\pi^*(a), \pi^*(b)) \in \pi(R)$
- $f \in \pi^*((a, b) : R)$ *iff* $(\pi^*(a), \pi^*(b)) \notin \pi(R)$

The interpretation of the subsumption relation given above was proposed by Patel-Schneider (23) for their four-valued semantics. Patel-Schneider's interpretation of the subsumption relation does not correspond to the material implication $\forall x[C(x) \rightarrow D(x)]$ in first-order logic. The latter is equivalent to $\forall x[\neg C(x) \vee D(x)]$ under the two-valued semantics, which corresponds to: "for every $o \in O$, $o \in \pi^*(C)^-$ or $o \in \pi^*(D)^+$" under the three-valued semantics. No conclusion can be drawn from $a : C$ and $C \sqsubseteq D$ under the three-valued semantics since there always exists an interpretation such that $\pi(a : C) = \{t, f\}$.

The entailment relation can be defined using the interpretations of propositions.

Definition 7 *Let $I = \langle O, \pi \rangle$ be an interpretation, let φ be a proposition, and let Σ be a set of propositions. The the entailment relation is defined as:*

- $I \models \varphi$ *iff* $t \in \pi^*(\varphi)$
- $I \models \Sigma$ *iff* $t \in \pi^*(\sigma)$ *for every* $\sigma \in \Sigma$.
- $\Sigma \models \varphi$ *iff* $I \models \Sigma$ *implies* $I \models \varphi$ *for each interpretation* I

Semantic tableaux We use a semantic tableaux method that is based on the semantic tableaux method for LP described by Bloesch (5). This tableaux method will enable us to identify the assumptions underlying relevant conflict minimal interpretations.

Bloesch proposes to label every proposition in the tableaux with either the labels $\mathbb{T}$ (at least true), $\mathbb{F}$ (at least false), or their complements $\overline{\mathbb{T}}$ and $\overline{\mathbb{F}}$, respectively. So, $\mathbb{T}\varphi$ corresponds to $t \in \pi(\varphi)$, $\overline{\mathbb{T}}\varphi$ corresponds to $t \notin \pi(\varphi)$, $\mathbb{F}\varphi$ corresponds to $f \in \pi(\varphi)$, and $\overline{\mathbb{F}}\varphi$ corresponds to $f \notin \pi(\varphi)$.

Although we do not need it in the semantic tableaux, we also make use of $\mathbb{C}\varphi$ and $\overline{\mathbb{C}}\varphi$, which corresponds semantically with $\{t, f\} = \pi(\varphi)$ and $\{t, f\} \neq \pi(\varphi)$, respectively. So, $\mathbb{C}\varphi$ is equivalent to: '$\mathbb{T}\varphi$ *and* $\mathbb{F}\varphi$', and $\overline{\mathbb{C}}\varphi$ is equivalent to: '$\overline{\mathbb{T}}\varphi$ *or* $\overline{\mathbb{F}}\varphi$'.

To prove that $\Sigma \models \varphi$ using Bloesch's tableaux method (5), we have to show that a tableaux with root $\Gamma = \{\mathbb{T}\sigma \mid \sigma \in \Sigma\} \cup \overline{\mathbb{T}}\varphi$ closes. The tableaux closes if every branch has a node in which for some proposition α the node contains: "$\mathbb{T}\alpha$ *and* $\overline{\mathbb{T}}\alpha$", or "$\mathbb{F}\alpha$ *and* $\overline{\mathbb{F}}\alpha$", or "$\overline{\mathbb{T}}\alpha$ *and* $\overline{\mathbb{F}}\alpha$".

Based on Bloesch's semantic tableaux method for LP, the following tableaux rules have been formulated. The soundness and completeness of the set of rules are easy to prove.

The individuals a and b in the tableaux rules must be existing individual names, while the individual x must be a new individual name.

$$\frac{\mathbb{T}\, a : \neg C}{\mathbb{F}\, a : C} \quad \frac{\overline{\mathbb{T}}\, a : \neg C}{\overline{\mathbb{F}}\, a : C} \quad \frac{\mathbb{F}\, a : \neg C}{\mathbb{T}\, a : C} \quad \frac{\overline{\mathbb{F}}\, a : \neg C}{\overline{\mathbb{F}}\, a : C}$$

$$\frac{\mathbb{T}\, a : C \sqcap D}{\mathbb{T}\, a : C, \mathbb{T}\, a : D} \quad \frac{\overline{\mathbb{T}}\, a : C \sqcap D}{\overline{\mathbb{T}}\, a : C \mid \overline{\mathbb{T}}\, a : D}$$

$$\frac{\mathbb{F}\, a : C \sqcap D}{\mathbb{F}\, a : C \mid \mathbb{F}\, a : D} \quad \frac{\overline{\mathbb{F}}\, a : C \sqcap D}{\overline{\mathbb{F}}\, a : C, \overline{\mathbb{F}}\, a : D}$$

$$\frac{\mathbb{T}\, a : C \sqcup D}{\mathbb{T}\, a : C \mid \mathbb{T}\, a : D} \quad \frac{\overline{\mathbb{T}}\, a : C \sqcup D}{\overline{\mathbb{T}}\, a : C, \overline{\mathbb{T}}\, a : D}$$

$$\frac{\mathbb{F}\, a : C \sqcup D}{\mathbb{F}\, a : C, \mathbb{F}\, a : D} \quad \frac{\overline{\mathbb{F}}\, a : C \sqcup D}{\overline{\mathbb{F}}\, a : C \mid \overline{\mathbb{F}}\, a : D}$$

$$\frac{\mathbb{T}\, a : \exists r.C}{\mathbb{T}\, (a, x) : r, \mathbb{T}\, x : C} \quad \frac{\overline{\mathbb{T}}\, a : \exists r.C, \mathbb{T}\, (a, b) : r}{\overline{\mathbb{T}}\, b : C}$$

$$\frac{\mathbb{F}\, a : \exists r.C, \mathbb{T}\, (a, b) : r}{\mathbb{F}\, b : C} \quad \frac{\overline{\mathbb{F}}\, a : \exists r.C}{\mathbb{T}\, (a, x) : r, \overline{\mathbb{F}}\, x : C}$$

$$\frac{\mathbb{T}\, a : \forall r.C, \mathbb{T}\, (a, b) : r}{\mathbb{T}\, b : C} \quad \frac{\overline{\mathbb{T}}\, a : \forall r.C}{\mathbb{T}\, (a, x) : r, \overline{\mathbb{T}}\, x : C}$$

$$\frac{\mathbb{F}\, a : \forall r.C}{\mathbb{T}\, (a, b) : r, \mathbb{F}\, b : C} \quad \frac{\overline{\mathbb{F}}\, a : \forall r.C, \mathbb{T}\, (a, b) : r}{\overline{\mathbb{F}}\, b : C}$$

$$\frac{}{\overline{\mathbb{T}}\, a : \top} \quad \frac{}{\overline{\mathbb{F}}\, a : \bot}$$

$$\frac{\mathbb{T}\, C \sqsubseteq D}{\overline{\mathbb{T}}\, a : C \mid \mathbb{T}\, a : D} \quad \frac{\mathbb{T}\, C \sqsubseteq D}{\overline{\mathbb{F}}\, a : D \mid \mathbb{F}\, a : C}$$

$$\frac{\overline{\mathbb{T}}\, C \sqsubseteq D}{\mathbb{T}\, x : C, \overline{\mathbb{T}}\, x : D \mid \mathbb{F}\, x : D, \overline{\mathbb{F}}\, x : C}$$

$$\frac{\mathbb{T}\, C = D}{\mathbb{T}\, C \sqsubseteq D, \mathbb{T}\, D \sqsubseteq C} \quad \frac{\overline{\mathbb{T}}\, C = D}{\overline{\mathbb{T}}\, C \sqsubseteq D \mid \overline{\mathbb{T}}\, D \sqsubseteq C}$$

An important issue is guaranteeing that the constructed semantic tableaux is always finite. The *blocking* method described by (9; 2) is used to guarantee the construction of a finite tableaux. A rule that is *blocked*, may not be not be used in the construction of the tableaux.

Definition 8 *Let Γ be a node of the tableau, and let x and y be two individual names. Moreover, let $\Gamma(x) = \{\mathbb{L}x : C \mid \mathbb{L}x : C \in \Gamma\}$.*

- $x <_r y$ *if* $(x, y) : R \in \Gamma$ *for some* $R \in \mathbf{R}$.
- y *is blocked if there is an individual name x such that: $x <_r^+ y$ and $\Gamma(y) \subseteq \Gamma(x)$, or $x <_r y$ and x is blocked.*

Conflict Minimal Interpretations A price that we pay for changing to the three-valued LP-semantics in order to handle inconsistencies is a reduction in the set of entailed conclusions, even if the knowledge and information is consistent.

Example 1 *The set of propositions $\Sigma = \{a : \neg C, a : C \sqcup D\}$ does not entail $a : D$ because there exists an interpretation $I = \langle O, \pi \rangle$ for Σ such that $\pi(a : C) = \{t, f\}$ and $\pi(a : D) = \{f\}$.*

Priest (26; 27) points out that more useful conclusions can be derived from the paraconsistent logic LP if we would prefer conflict-minimal interpretations. The resulting logic is LPm. Here we follow the same approach. First, we define a conflict ordering on interpretations.

Definition 9 *Let $\mathbf{C}$ be a set of atomic concepts, let $\mathbf{N}$ be a set of individual names, and let I_1 and I_2 be two three-valued interpretations.*

The interpretation I_1 contains less conflicts than the interpretation I_2, denoted by $I_1 <_c I_2$, iff:

$$\{a : C \mid a \in \mathbf{N}, C \in \mathbf{C}, \pi_1(a : C) = \{t, f\}\} \subset \{a : C \mid a \in \mathbf{N}, C \in \mathbf{C}, \pi_2(a : C) = \{t, f\}\}$$

The following example gives an illustration of a conflict ordering for the set of propositions of Example 1.

Example 2 *Let $\Sigma = \{a : \neg C, a : C \sqcup D\}$ be a set of propositions and let I_1, I_2, I_3, I_4 and I_5 be five interpretations such that:*

- $\pi^*_1(a : C) = \{f\}$, $\pi^*_1(a : D) = \{t\}$,
- $\pi^*_2(a : C) = \{f\}$, $\pi^*_2(a : D) = \{t, f\}$.
- $\pi^*_3(a : C) = \{t, f\}$, $\pi^*_3(a : D) = \{t\}$,
- $\pi^*_4(a : C) = \{t, f\}$, $\pi^*_4(a : D) = \{f\}$,
- $\pi^*_5(a : C) = \{t, f\}$, $\pi^*_5(a : D) = \{t, f\}$.

Then $I_1 <_c I_2$, $I_1 <_c I_3$, $I_1 <_c I_4$, $I_1 <_c I_5$, $I_2 <_c I_5$, $I_3 <_c I_5$ and $I_4 <_c I_5$.

Using the conflict ordering, we define the conflict minimal interpretations.

Definition 10 *Let I_1 be a three-valued interpretation and let Σ be a set of propositions.*

I_1 is a conflict minimal interpretation of Σ, denoted by $I_1 \models_{<_c} \Sigma$, iff $I_1 \models \Sigma$ and for no interpretation I_2 such that $I_2 <_c I_1$, $I_2 \models \Sigma$ holds.

In Example 2, I_1 is the only conflict-minimal interpretation.

The conflict-minimal entailment of a proposition by a set of propositions can now be defined.

Definition 11 *Let $\Sigma = (\mathcal{T} \cup \mathcal{A})$ be a set of propositions and let φ be a proposition.*

Σ entails conflict-minimally the proposition φ, denoted by $\Sigma \models_{<_c} \varphi$, iff for every interpretation I, if $I \models_{<_c} \Sigma$, then $I \models \varphi$.

The conflict-minimal interpretations in Example 2 entail the conclusion $a : D$.

The subsumption relation The conflict-minimal interpretations enables us to use an interpretation of the subsumption relation based on the material implication.

- For every $o \in O$, $o \in \pi^*(C)^-$ or $o \in \pi^*(D)^+$

This semantics of the subsumption relation resolves a problem with the semantics of Patel-Schneider (23). Under Patel-Schneider's semantics, $\{a : C, a : \neg C, C \sqsubseteq D\}$ entails $a : D$. This entailment is undesirable if information about $a : C$ is contradictory.

The tableaux rules of the new interpretation are:

$$\frac{\mathbb{T}\, C \sqsubseteq D}{\mathbb{F}\, a : C \mid \mathbb{T}\, a : D} \qquad \frac{\overline{\mathbb{T}}\, C \sqsubseteq D}{\mathbb{T}\, a : C, \mathbb{F}\, a : D}$$

Arguments for conclusions supported by conflict minimal interpretations

The conflict-minimal interpretations of a knowledge base entail more useful conclusions. Unfortunately, focusing on conclusions supported by conflict-minimal interpretations makes the reasoning process *non-monotonic*. Adding the assertion $a : \neg D$ to the set of propositions in Example 2 eliminates interpretations I_1 and I_3, which includes the only conflict-minimal interpretation I_1. The interpretations I_2 and I_4 are the new conflict-minimal interpretations. Unlike the original conflict-minimal interpretation I_1, the new conflict-minimal interpretations I_2 and I_4 do not entail $a : D$.

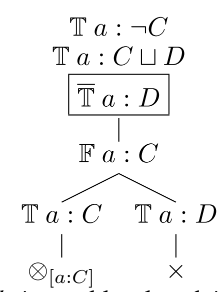
Deriving conclusions supported by the conflict-minimal interpretations is problematic because of the non-monotonicity. The modern way to deal with non-monotonicity is by giving an argument supporting a conclusion, and subsequently verifying whether there are no counter-arguments (12). Here we will follow this argumentation-based approach.

We propose an approach for deriving arguments that uses the semantic tableaux method for our paraconsistent logic as a starting point. The approach is based on the observation that an interpretation satisfying the root of a semantic tableaux will also satisfy one of the leafs. Now suppose that the only leafs of a tableaux that are not closed; i.e., leaf in which we do not have "$\mathbb{T}\alpha$ *and* $\overline{\mathbb{T}}\alpha$" or "$\mathbb{F}\alpha$ *and* $\overline{\mathbb{F}}\alpha$" or "$\overline{\mathbb{T}}\alpha$ *and* $\overline{\mathbb{F}}\alpha$", are leafs in which "$\mathbb{T}\alpha$ *and* $\mathbb{F}\alpha$" holds for some proposition α. So, in every open branch of the tableaux, $\mathbb{C}\alpha$ holds for some proposition α. If we can assume that there are no conflicts w.r.t. each proposition α in the conflict-minimal interpretations, then we can also close the open branches. The set of assumptions $\overline{\mathbb{C}}\alpha$, equivalent to "$\overline{\mathbb{T}}\alpha$ *or* $\overline{\mathbb{F}}\alpha$", that we need to close the open branches, will be used as the argument for the conclusion supported by the semantic tableaux.

An advantage of the proposed approach is that there is no need to consider arguments if a conclusion already holds without considering conflict-minimal interpretations.

A branch that can be closed *assuming* that the conflict-minimal interpretations contain **no** conflicts with respect to the proposition α; i.e., assuming $\overline{\mathbb{C}}\alpha$, will be called a *weakly closed* branch. We will call a tableaux *weakly closed* if some branches are weakly closed and all other branches are closed. If we can (weakly) close a tableaux for $\Gamma = \{\mathbb{T}\sigma \mid \sigma \in (\mathcal{T} \cup \mathcal{A})\} \cup \overline{\mathbb{T}}\varphi$, we consider the set of assumptions $\overline{\mathbb{C}}\alpha$ needed to weakly close the tableaux, to be the argument supporting $\Sigma \models_{<_c} \varphi$. Example 3 gives an illustration.

Example 3 *Let $\Sigma = \{a : \neg C, a : C \sqcup D\}$ be a set of propositions. To verify whether $a : D$ holds, we may construct the following tableaux:*



Only the left branch is weakly closed *in this tableau. We assume that the assertion $a : C$ will* not *be assigned* CONFLICT *in any conflict-minimal interpretation. That is, we assume that $\overline{\mathbb{C}}\, a : C$ holds.*

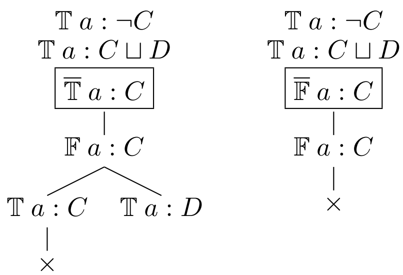
In the following definition of an argument, we consider arguments for $\mathbb{T}\varphi$ and $\mathbb{F}\varphi$.

Definition 12 *Let Σ be a set of propositions and let φ a proposition. Moreover, let $\mathscr{T}$ be a (weakly) closed semantic tableaux with root $\Gamma = \{\mathbb{T}\sigma \mid \sigma \in \Sigma\} \cup \overline{\mathbb{L}}\varphi$ and $\mathbb{L} \in \{\mathbb{T}, \mathbb{F}\}$. Finally, let $\{\overline{\mathbb{C}}\alpha_1, \ldots, \overline{\mathbb{C}}\alpha_k\}$ be the set of assumptions on which the closures of weakly closed branches are based.*

Then $A = (\{\overline{\mathbb{C}}\alpha_1, \ldots, \overline{\mathbb{C}}\alpha_k\}, \mathbb{L}\varphi)$ is an argument.

The next step is to verify whether the assumptions: $\overline{\mathbb{C}}\alpha_i$ are valid. If one of the assumptions does not hold, we have a *counter-argument* for our argument supporting $\Sigma \models_{<_c} \varphi$. To verify the correctness of an assumption, we add the assumption to Σ. Since an assumption $\overline{\mathbb{C}}\alpha$ is equivalent to: "$\overline{\mathbb{T}}\alpha$ *or* $\overline{\mathbb{F}}\alpha$", we can consider $\overline{\mathbb{T}}\alpha$ and $\overline{\mathbb{F}}\alpha$ separately. Example 4 gives an illustration for the assumption $\overline{\mathbb{C}}\ a : C$ used in Example 3.

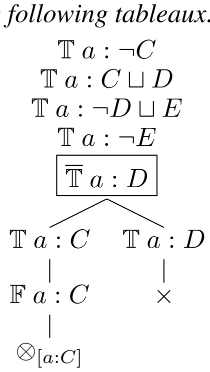
Example 4 *Let $\Sigma = \{a : \neg C, a : C \sqcup D\}$ be a set of propositions. To verify whether the assumption $\overline{\mathbb{C}}\ a : C$ holds in every conflict minimal interpretation, we may construct a tableaux assuming $\overline{\mathbb{T}}\ a : C$ and a tableaux assuming $\overline{\mathbb{F}}\ a : C$:*



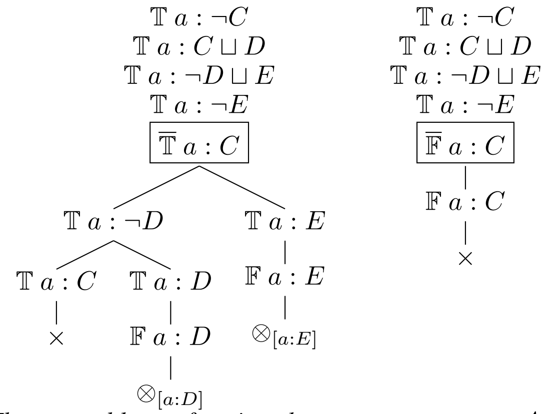
The right branch of the first tableaux cannot be closed. Therefore, the assumption $\overline{\mathbb{T}}\ a : C$ is valid, implying that the assumption $\overline{\mathbb{C}}\ a : C$ is also valid. Hence, there exists no *counter-argument.*

Since the validity of assumptions must be verified with respect to conflict-minimal interpretations, assumptions may also be used in the counter-arguments. This implies that we may have to verify whether there exists a counter-argument for a counter-argument. Example 5 gives an illustration.

Example 5 *Let $\Sigma = \{a : \neg C, a : C \sqcup D, a : \neg D \sqcup E, a : \neg E\}$ be a set of propositions. To verify whether $a : D$ holds, we may construct the following tableaux:*



This weakly closed tableaux implies the argument $A_0 = (\{\overline{\mathbb{C}}\ a : C\}, \mathbb{T}\ a : D)$. Next, we have to verify whether there exists a counter-argument for A_0. To verify the existence of a counter-argument, we construct two tableaux, one for $\overline{\mathbb{T}}\ a : C$ and one for $\overline{\mathbb{F}}\ a : C$. As we can see below, both tableaux are (weakly)-closed, and therefore form the counter-argument $A_1 = (\{\overline{\mathbb{C}}\ a : D, \overline{\mathbb{C}}\ a : E\}, \mathbb{C}\ a : C)$. We say that the argument A_1 attacks *the argument A_0 because the former is a counter-argument of the latter.*



The two tableaux forming the counter-argument A_1 are closed under the assumptions: $\overline{\mathbb{C}}\ a : D$ and $\overline{\mathbb{C}}\ a : E$. So, A_1 is a valid argument if there exists no valid counter-argument for $\mathbb{C}\ a : D$, and no counter-argument for $\mathbb{C}\ a : E$.

Argument A_1 is equivalent to two other arguments, namely: $A_2 = (\{\overline{\mathbb{C}}\ a : C, \overline{\mathbb{C}}\ a : E\}, \mathbb{C}\ a : D)$ and $A_3 = (\{\overline{\mathbb{C}}\ a : C, \overline{\mathbb{C}}\ a : D\}, \mathbb{C}\ a : E)$. A proof of the equivalence will be given in the next section, Proposition 1.

The arguments A_2 and A_3 implied by A_1 are both counter-arguments of A_1. Moreover, A_1 is a counter-argument of A_2 and A_3, and A_2 and A_3 are counter-arguments of each other. No other counter-arguments can be identified in this example. Figure 1 show all the arguments and the attack relation, denoted by the arrows, between the arguments.

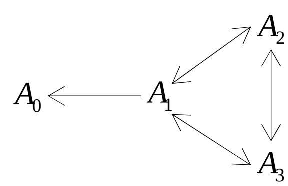


Figure 1: The attack relations between the arguments of Example 5.

We will now formally define the arguments and the attack relations that we can derive from the constructed semantic tableaux.

Definition 13 *Let Σ be a set of propositions and let $\overline{\mathbb{C}}\alpha$ ="$\overline{\mathbb{T}}\alpha$ or $\overline{\mathbb{F}}\alpha$" be an assumption in the argument A. Moreover, let $\mathscr{T}_1$ be a (weakly) closed semantic tableaux with root $\Gamma_1 = \{\mathbb{T}\sigma \mid \sigma \in \Sigma\} \cup \overline{\mathbb{T}}\alpha$ and let $\mathscr{T}_2$ be a (weakly) closed semantic tableaux with root $\Gamma_2 = \{\mathbb{T}\sigma \mid \sigma \in \Sigma\} \cup \overline{\mathbb{F}}\alpha$. Finally, let $\{\overline{\mathbb{C}}\alpha_1, \ldots, \overline{\mathbb{C}}\alpha_k\}$ be the set of assumptions on which the weakly closed branches in the tableaux $\mathscr{T}_1$ or the tableaux $\mathscr{T}_2$ are based.*

Then $A' = (\{\overline{\mathbb{C}}\alpha_1, \ldots, \overline{\mathbb{C}}\alpha_k\}, \mathbb{C}\alpha)$ is a counter-argument of the argument A. We say that the argument A' attacks *the argument A, denoted by: $A' \longrightarrow A$.*

The form of argumentation that we have here is called assumption-based argumentation (ABA), which has been developed since the end of the 1980's (6; 7; 11; 13; 30; 31).

Example 5 shows that an argument can be counter-argument of an argument and vice versa; e.g., arguments A_2 and A_3. This raises the question which arguments are valid. Argumentation theory and especially the argumentation framework (AF) introduced by Dung (12) provides an answer.

Arguments are viewed in an argumentation framework as atoms over which an attack relation is defined. Figure 1 shows the arguments and the attack relations between the arguments forming the argumentation framework of Example 5. The formal specification of an argumentation framework is given by the next definition.

Definition 14 *An argumentation framework is a couple $AF = (\mathscr{A}, \longrightarrow)$ where $\mathscr{A}$ is a finite set of arguments and $\longrightarrow \subseteq \mathscr{A} \times \mathscr{A}$ is an attack relation over the arguments.*

For convenience, we extend the attack relation $\longrightarrow$ to sets of arguments.

Definition 15 *Let $A \in \mathscr{A}$ be an argument and let $\mathscr{S}, \mathscr{P} \subseteq \mathscr{A}$ be two sets of arguments. We define:*

- *$\mathscr{S} \longrightarrow A$ iff for some $B \in \mathscr{S}$, $B \longrightarrow A$.*
- *$A \longrightarrow \mathscr{S}$ iff for some $B \in \mathscr{S}$, $A \longrightarrow B$.*
- *$\mathscr{S} \longrightarrow \mathscr{P}$ iff for some $B \in \mathscr{S}$ and $C \in \mathscr{P}$, $B \longrightarrow C$.*

Dung (12) describes different argumentation semantics for an argumentation framework in terms of sets of acceptable arguments. These semantics are based on the idea of selecting a coherent subset $\mathscr{E}$ of the set of arguments $\mathscr{A}$ of the argumentation framework $AF = (\mathscr{A}, \longrightarrow)$. Such a set of arguments $\mathscr{E}$ is called an *argument extension*. The arguments of an argument extension support propositions that give a coherent description of what might hold in the world. Clearly, a basic requirement of an argument extension is being *conflict-free*; i.e., no argument in an argument extension attacks another argument in the argument extension. Besides being conflict-free, an argument extension should defend itself against attacking arguments by attacking the attacker.

Definition 16 *Let $AF = (\mathscr{A}, \longrightarrow)$ be an argumentation framework and let $\mathscr{S} \subseteq \mathscr{A}$ be a set of arguments.*

- *$\mathscr{S}$ is* conflict-free *iff $\mathscr{S} \not\longrightarrow \mathscr{S}$.*
- *$\mathscr{S}$ defends an argument $A \in \mathscr{A}$ iff for every argument $B \in \mathscr{A}$ such that $B \longrightarrow A$, $\mathscr{S} \longrightarrow B$.*

Not every conflict-free set of arguments that defends itself, is considered to be an argument extension. Several additional requirements have been formulated by Dung (12), resulting in three different semantics: the *stable*, the *preferred* and the *grounded* semantics.

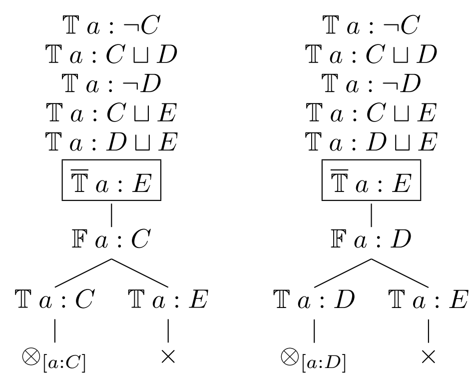
Definition 17 *Let $AF = (\mathscr{A}, \longrightarrow)$ be an argumentation framework and let $\mathscr{E} \subseteq \mathscr{A}$.*

- *$\mathscr{E}$ is a* stable extension *iff $\mathscr{E}$ is conflict-free,and for every argument $A \in (\mathscr{A} \setminus \mathscr{E})$, $\mathscr{E} \longrightarrow A$; i.e., $\mathscr{E}$ defends itself against every possible attack by arguments in $\mathscr{A} \setminus \mathscr{E}$.*
- *$\mathscr{E}$ is a* preferred extension *iff $\mathscr{E}$ is maximal (w.r.t. $\subseteq$) set of arguments that (1) is conflict-free,and (2) $\mathscr{E}$ defends every argument $A \in \mathscr{E}$.*
- *$\mathscr{E}$ is a* grounded extension *iff $\mathscr{E}$ is the minimal (w.r.t. $\subseteq$) set of arguments that (1) is conflict-free, (2) defends every argument $A \in \mathscr{E}$, and (3) contains all arguments in $\mathscr{A}$ it defends.*

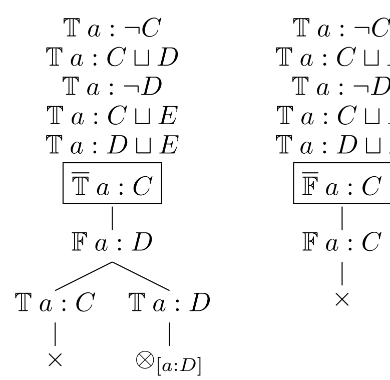
We are interested in stable semantics. We will show in the next section that stable extensions correspond to conflict-minimal interpretations. More specifically, we will prove that a conclusion supported by an argument in every stable extension, is entailed by every conflict-minimal interpretation, and vice versa.

Is it possible that a conclusion is supported by a different argument in every stable extension? The answer is Yes, as is illustrated by Example 6. In this example we have two arguments supporting the conclusion $a : E$, namely A_0 and A_1. As can be seen in Figure 2, there are two stable extensions of the argumentation framework. One extension contains the argument A_0 and the other contains the argument A_1. So, in every extension there is an argument supporting the conclusion $a : E$. Hence, $\Sigma \models_{<_c} a : E$.

Example 6 *Let $\Sigma = \{a : \neg C, a : C \sqcup D, a : \neg D, a : C \sqcup E, a : D \sqcup E\}$ be a set of propositions. The following two tableaux imply the two arguments $A_0 = (\{\overline{\mathbb{C}}\ a : C\}, \mathbb{T}\ a : E)$ and $A_1 = (\{\overline{\mathbb{C}}\ a : D\}, \mathbb{T}\ a : E)$, both supporting the conclusion $a : E$:*



The assumption $\overline{\mathbb{C}}\ a : C$ in argument A_0 makes it possible to determine a counter-argument $A_2 = (\{\overline{\mathbb{C}}\ a : D\}, \mathbb{C}\ a : C)$ using of the following two tableaux:



According to Proposition 1, A_2 implies the counter-argument $A_3 = (\{\overline{\mathbb{C}}\ a : C\}, \mathbb{C}\ a : D)$ of A_1 and A_2. A_2 is also a counter-argument of A_3. Figure 2 shows the attack relations between the arguments A_0, A_1, A_2 and A_3.

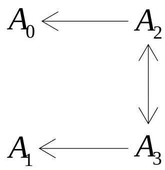


Figure 2: The attack relations between the arguments of Example 6.

Example 7 gives an illustration of the semantic interpretations of Example 6. The example shows two conflict-minimal interpretations. These conflict-minimal interpretations correspond with the two *stable extensions*. Interpretation I_1 entails $a : E$ because I_1 must entail $a : C \sqcup E$ and I_1 does not entail $a : C$, and interpretation I_2 entails $a : E$ because I_2 must entail $a : D \sqcup E$ and I_2 does not entail $a : D$.

Example 7 *Let $\Sigma = \{a : \neg C, a : C \sqcup D, a : \neg D, a : C \sqcup E, a : D \sqcup E\}$ be a set of propositions. There are two conflict-minimal interpretations containing the following interpretation functions:*

- $\pi_1(a : C) = \{f\}$, $\pi_1(a : D) = \{t, f\}$, $\pi_1(a : E) = \{t\}$.
- $\pi_2(a : C) = \{t, f\}$, $\pi_2(a : D) = \{f\}$, $\pi_2(a : E) = \{t\}$.

In both interpretations $a : E$ is entailed.

Correctness and completeness proofs

In this section we investigate whether the proposed approach is correct. That is whether a proposition supported by an argument in every stable extension is entailed by every conflict-minimal interpretation. Moreover, we investigate whether the approach is complete. That is, whether a proposition entailed by every conflict-minimal interpretation is supported by an argument in every stable extension.

In the following theorem we will use the notion of "a complete set of arguments relevant to φ". This set of arguments $\mathscr{A}$ consists of all argument A supporting φ, all possible counter-arguments, all possible counter arguments of the counter-arguments, etc.

Definition 18 *A complete set of arguments $\mathscr{A}$ relevant to φ satisfies the following requirements:*

- $\{A \mid A$ *supports* $\varphi\} \subseteq \mathscr{A}$.
- *If $A \in \mathscr{A}$ and B is a counter-argument of A that we can derive, then $B \in \mathscr{A}$ and $(B, A) \in \longrightarrow$.*

Theorem 1 (correctness and completeness) *Let Σ be a set of propositions and let φ be a proposition. Moreover, let $\mathscr{A}$ be a complete set of arguments relevant to φ, let $\longrightarrow \subseteq \mathscr{A} \times \mathscr{A}$ be the attack relation determined by $\mathscr{A}$, and let $(\mathscr{A}, \longrightarrow)$ be the argumentation framework. Finally, let $\mathscr{E}_1, \ldots, \mathscr{E}_k$ be all stable extensions of the argumentation framework $(\mathscr{A}, \longrightarrow)$.*

Σ entails the proposition φ using the conflict-minimal three-valued semantics; i.e., $\Sigma \models_{<_c} \varphi$, iff φ is supported by an argument in every stable extension $\mathscr{E}_i$ of $(\mathscr{A}, \longrightarrow)$.

To prove Theorem 1, we need the following lemmas. In these lemmas we will use the following notations: We will use $I \models \mathbb{T}\alpha$ to denote that $t \in I(\alpha)$ ($I \models \alpha$), and $I \models \mathbb{F}\alpha$ to denote that $f \in I(\alpha)$. Moreover, we will use $\Sigma \models \mathbb{T}\alpha$ and $\Sigma \models \mathbb{F}\alpha$ to denote that $\mathbb{T}\alpha$ and $\mathbb{F}\alpha$, respectively, hold in all three-valued interpretations of Σ.

The first lemma proves the correctness of the arguments in $\mathscr{A}$.

Lemma 1 (correctness of arguments) *Let Σ be a set of propositions and let φ be a proposition. Moreover, let $\mathbb{L}$ be either the label $\mathbb{T}$ or $\mathbb{F}$.*

If a semantic tableaux with root $\Gamma = \{\mathbb{T}\sigma \mid \sigma \in \Sigma\} \cup \{\overline{\mathbb{L}}\varphi\}$ is weakly closed, and if $\{\overline{\mathbb{C}}\alpha_1, \ldots, \overline{\mathbb{C}}\alpha_k\}$ is the set of weak closure assumptions implied by all the weakly closed leafs of the tableaux, then

$$\{\overline{\mathbb{C}}\alpha_1, \ldots, \overline{\mathbb{C}}\alpha_k\} \cup \{\mathbb{T}\sigma \mid \sigma \in \Sigma\} \models \mathbb{L}\varphi$$

Proof Suppose that $\{\overline{\mathbb{C}}\alpha_1, \ldots, \overline{\mathbb{C}}\alpha_k\} \cup \{\mathbb{T}\sigma \mid \sigma \in \Sigma\} \not\models \mathbb{L}\varphi$. Then there must be an interpretation I satisfying $\{\overline{\mathbb{C}}\alpha_1, \ldots, \overline{\mathbb{C}}\alpha_k\} \cup \{\mathbb{T}\sigma \mid \sigma \in \Sigma\}$ but not $\mathbb{L}\varphi$. So, $I \models \{\overline{\mathbb{C}}\alpha_1, \ldots, \overline{\mathbb{C}}\alpha_k\} \cup \{\mathbb{T}\sigma \mid \sigma \in \Sigma\} \cup \{\overline{\mathbb{L}}\varphi\}$. We can create a tableaux for $\{\overline{\mathbb{C}}\alpha_1, \ldots, \overline{\mathbb{C}}\alpha_k\} \cup \{\mathbb{T}\sigma \mid \sigma \in \Sigma\} \cup \{\overline{\mathbb{L}}\varphi\}$ by adding the assumptions $\{\overline{\mathbb{C}}\alpha_1, \ldots, \overline{\mathbb{C}}\alpha_k\}$ to every node in the original tableaux with root Γ. Let $\Gamma^* = \{\overline{\mathbb{C}}\alpha_1, \ldots, \overline{\mathbb{C}}\alpha_k\} \cup \{\mathbb{T}\sigma \mid \sigma \in \Sigma\} \cup \{\overline{\mathbb{L}}\varphi\}$ be the root of the resulting tableaux. Since $I \models \Gamma^*$, there must be a leaf Λ^* of the new tableaux and $I \models \Lambda^*$. The corresponding leaf Λ in the original tableaux with root Γ is either strongly or weakly closed.

– If Λ is strongly closed, then so is Λ^* and we have a contradiction.

– If Λ is weakly closed, then the weak closure implies one of the assumptions $\overline{\mathbb{C}}\alpha_i$ because $\{\mathbb{T}\alpha_i, \mathbb{F}\alpha_i\} \subseteq \Lambda$. Therefore, $\{\mathbb{T}\alpha_i, \mathbb{F}\alpha_i\} \subseteq \Lambda^*$. Since $\{\mathbb{T}\alpha_i, \mathbb{F}\alpha_i\}$ implies $\mathbb{C}\alpha_i$ and since $\overline{\mathbb{C}}\alpha_i \in \Lambda^*$, $I \not\models \Lambda^*$ The latter contradicts with $I \models \Lambda^*$.

Hence, the lemma holds. □

The above lemma implies that the assumptions of an argument $A = (\{\overline{\mathbb{C}}\alpha_1, \ldots, \overline{\mathbb{C}}\alpha_k\}, \mathbb{L}\varphi)$ together with Σ entail the conclusion of A.

The next lemma proves the completeness of the set of arguments $\mathscr{A}$.

Lemma 2 (completeness of arguments) *Let Σ be a set of propositions and let φ be a proposition. Moreover, let $\mathbb{L}$ be either the label $\mathbb{T}$ or $\mathbb{F}$.*

If $\{\overline{\mathbb{C}}\alpha_1, \ldots, \overline{\mathbb{C}}\alpha_k\}$ is a set of atomic assumptions with $\alpha_i = a_i : C_i$, $a_i \in \mathbf{N}$ and $C_i \in \mathbf{C_i}$, and if

$$\{\overline{\mathbb{C}}\alpha_1, \ldots, \overline{\mathbb{C}}\alpha_k\} \cup \{\mathbb{T}\sigma \mid \sigma \in \Sigma\} \models \mathbb{L}\varphi$$

then there is a semantic tableaux with root $\Gamma = \{\mathbb{T}\sigma \mid \sigma \in \Sigma\} \cup \{\overline{\mathbb{L}}\varphi\}$, and the tableaux is weakly closed.

Proof Let $\Gamma = \{\mathbb{T}\sigma \mid \sigma \in \Sigma\} \cup \{\overline{\mathbb{L}}\varphi\}$ be the root of a semantic tableaux.

Suppose that the tableaux is *not* weakly closed. Then there is an open leaf Λ. We can create a tableaux for $\{\overline{\mathbb{C}}\alpha_1, \ldots, \overline{\mathbb{C}}\alpha_k\} \cup \{\mathbb{T}\sigma \mid \sigma \in \Sigma\} \cup \{\overline{\mathbb{L}}\varphi\}$ by adding the assumptions $\{\overline{\mathbb{C}}\alpha_1, \ldots, \overline{\mathbb{C}}\alpha_k\}$ to every node in the original tableaux with root Γ. Let $\Gamma^* = \{\overline{\mathbb{C}}\alpha_1, \ldots, \overline{\mathbb{C}}\alpha_k\} \cup \{\mathbb{T}\sigma \mid \sigma \in \Sigma\} \cup \{\overline{\mathbb{L}}\varphi\}$ be the root of the resulting tableaux. Since $\{\overline{\mathbb{C}}\alpha_1, \ldots, \overline{\mathbb{C}}\alpha_k\} \cup \{\mathbb{T}\sigma \mid \sigma \in \Sigma\} \models \mathbb{L}\varphi$, there exists no interpretation I such that $I \models \Gamma^*$. Therefore, there exists no interpretation I such that $I \models \Lambda^*$. Since we considered only atomic assumptions $\overline{\mathbb{C}}\alpha_i$, we cannot extend the tableaux by rewriting a proposition in Λ^*. Therefore, Λ^* must be strongly closed and for some α_i, $\{\mathbb{T}\alpha_i, \mathbb{F}\alpha_i\} \subseteq \Lambda^*$. This implies that $\{\mathbb{T}\alpha_i, \mathbb{F}\alpha_i\} \subseteq \Lambda$. Hence, Λ is weakly closed under the assumption $\overline{\mathbb{C}}\alpha_i$. Contradiction.

Hence, the lemma holds. □

The above lemma implies that we can find an argument $A = (\{\overline{\mathbb{C}}\alpha_1, \ldots, \overline{\mathbb{C}}\alpha_k\}, \mathbb{L}\varphi)$ for any set of assumption that, together with Σ, entails a conclusion $\mathbb{L}\varphi$.

The following lemma proves that for every conflict $\mathbb{C}\varphi$ entailed by a conflict-minimal interpretation, we can find an argument supporting $\mathbb{C}\varphi$ of which the assumptions are entailed by the conflict-minimal interpretation.

Lemma 3 *Let Σ be a set of propositions and let $I = \langle O, \pi \rangle$ be a conflict-minimal interpretation of Σ. Moreover, let φ be a proposition.*

If $I \models \mathbb{C}\varphi$ holds, then there is an argument $A = (\{\overline{\mathbb{C}}\alpha_1, \ldots, \overline{\mathbb{C}}\alpha_k\}, \mathbb{C}\varphi)$ supporting $\mathbb{C}\varphi$ and for every assumption $\overline{\mathbb{C}}\alpha_i$, $I \models \overline{\mathbb{C}}\alpha_i$ holds.

Proof Let I be a conflict-minimal interpretation of Σ.

Suppose that $I \models \mathbb{C}\varphi$ holds. We can construct a tableaux for:

$$\begin{aligned}\Gamma \;=\; & \{\mathbb{T}\sigma \mid \sigma \in \Sigma\} \cup \{\overline{\mathbb{C}}\varphi\} \cup \\ & \{\overline{\mathbb{C}}\; a : C \mid C \in \mathbf{C}, \pi(a : C) \neq \{t, f\}\}\end{aligned}$$

Suppose that this tableaux is not strongly closed. Then there is an interpretation $I' = \langle O, \pi' \rangle$ satisfying the root Γ. Clearly, $I' <_c I$ because for every $a : C$ with $C \in \mathbf{C}$, if $\pi(a : C) \neq \{t, f\}$, then $\pi'(a : C) \neq \{t, f\}$. Since I is a conflict-minimal interpretation and since $I' \not\models \mathbb{C}\varphi$, we have a contradiction.

Hence, the tableaux is closed.

Since the tableaux with root Γ is closed, we can identify all assertions in $\{\overline{\mathbb{C}}\; a : C \mid C \in \mathbf{C}, \pi(a : C) \neq \{t, f\}\}$ that are **not** used to close a leaf of the tableaux. These assertions $\overline{\mathbb{C}}\; a : C$ play no role in the construction of the tableaux and can therefore be removed from every node of the tableaux. The result is still a valid and closed semantic tableaux with a new root Γ'. The assertions in $\{\overline{\mathbb{C}}\; a : C \mid C \in \mathbf{C}, \pi(a : C) \neq \{t, f\}\} \cap \Gamma'$ must all be used to strongly close leafs of the tableaux Γ', and also of Γ. A leaf that is strongly closed because of $\overline{\mathbb{C}}\; a : C$ can be closed weakly under the assumption $\overline{\mathbb{C}}\; a : C$. So, we may remove the remaining assertions $\overline{\mathbb{C}}\; a : C$ from the root Γ'. The result is still a valid semantic tableaux with root $\Gamma'' = \{\mathbb{T}\sigma \mid \sigma \in \Sigma\} \cup \{\overline{\mathbb{C}}\varphi\}$. This tableaux with root Γ'' is weakly closed, and by the construction of the tableaux, $I \models \overline{\mathbb{C}}\; a : C$ holds for every assumption $\overline{\mathbb{C}}\; a : C$ implied by a weak closure. Hence, we have constructed an argument $A = (\{\overline{\mathbb{C}}\alpha_1, \ldots, \overline{\mathbb{C}}\alpha_k\}, \mathbb{C}\varphi)$ supporting $\mathbb{C}\varphi$ and for every assumption $\overline{\mathbb{C}}\alpha_i$, $I \models \overline{\mathbb{C}}\alpha_i$ holds.

Hence, the lemma holds. □

For the next lemma we need the following definition of a set of assumptions that is allowed by an extension.

Definition 19 *Let Ω be the set of all assumptions $\overline{\mathbb{C}}\alpha$ in the arguments $\mathscr{A}$. For any extension $\mathscr{E} \subseteq \mathscr{A}$,*

$$\Omega(\mathscr{E}) = \{\overline{\mathbb{C}}\alpha \in \Omega \mid \textit{no argument } A \in \mathscr{E} \textit{ supports } \mathbb{C}\alpha\}$$

is the set of assumptions allowed by the extension $\mathscr{E}$.

The last lemma proves that for every conflict-minimal interpretation there is a corresponding stable extension.

Lemma 4 *Let Σ be a set of propositions and let φ be a proposition. Moreover, let $\mathscr{A}$ be the complete set of arguments relevant to φ, let $\longrightarrow \subseteq \mathscr{A} \times \mathscr{A}$ be the attack relation determined by $\mathscr{A}$, and let $(\mathscr{A}, \longrightarrow)$ be the argumentation framework.*

For every conflict-minimal interpretation I of Σ, there is a stable extension $\mathscr{E}$ of $(\mathscr{A}, \longrightarrow)$ such that $I \models \Omega(\mathscr{E})$.

Proof Let I be a conflict-minimal interpretation and let

$$\mathscr{E} = \{(\{\overline{\mathbb{C}}\alpha_1, \ldots, \overline{\mathbb{C}}\alpha_k\}, \varphi) \in \mathscr{A} \mid I \models \{\overline{\mathbb{C}}\alpha_1, \ldots, \overline{\mathbb{C}}\alpha_k\}\}$$

be the set of arguments $A = (\{\overline{\mathbb{C}}\alpha_1, \ldots, \overline{\mathbb{C}}\alpha_k\}, \varphi)$ of which the assumptions are entailed by I.

Suppose $\mathscr{E}$ is not conflict-free. Then there is an argument $B \in \mathscr{E}$ such that $B \longrightarrow A$ with $A \in \mathscr{E}$. So, B supports $\mathbb{C}\psi$ and $\overline{\mathbb{C}}\psi$ is an assumption of A. Since I entails the assumptions of A, $I \not\models \mathbb{C}\psi$. Since I is a conflict-minimal interpretation of Σ entailing the assumptions of B, according to Lemma 1, $I \models \mathbb{C}\psi$. Contradiction.

Hence, $\mathscr{E}$ is a conflict-free set of argument.

Suppose that there exists an argument $A \in \mathscr{A}$ such that $A \notin \mathscr{E}$. Then, for some assumption $\overline{\mathbb{C}}\alpha$ of A, $I \not\models \overline{\mathbb{C}}\alpha$. So, $I \models \mathbb{C}\alpha$, and according to Lemma 3, there is an argument $B \in \mathscr{E}$ supporting $\mathbb{C}\alpha$. Therefore, $B \longrightarrow A$.

Hence, $\mathscr{E}$ attacks every argument $A \in \mathscr{A} \backslash \mathscr{E}$. Since $\mathscr{E}$ is also conflict-free, $\mathscr{E}$ is a *stable* extension of $(\mathscr{A}, \longrightarrow)$.

Suppose that $I \not\models \Omega(\mathscr{E})$. Then there is a $\overline{\mathbb{C}}\alpha \in \Omega(\mathscr{E})$ and $I \models \mathbb{C}\alpha$. According to Lemma 3, there is an argument $A = (\{\overline{\mathbb{C}}\alpha_1, \ldots, \overline{\mathbb{C}}\alpha_k\}, \mathbb{C}\alpha)$ and $I \models \{\overline{\mathbb{C}}\alpha_1, \ldots, \overline{\mathbb{C}}\alpha_k\}$. So, $A \in \mathscr{E}$ and therefore, $\overline{\mathbb{C}}\alpha \notin \Omega(\mathscr{E})$. Contradiction.

Hence, $I \models \Omega(\mathscr{E})$. □

Using the results of the above lemmas, we can now prove the theorem.

Proof of Theorem 1
($\Rightarrow$) Let $\Sigma \models_{<_c} \varphi$.

Suppose that there is stable extension $\mathscr{E}_i$ that does not contain an argument for φ. Then according to Lemma 2, $\{\mathbb{T}\sigma \mid \sigma \in \Sigma\} \cup \Omega(\mathscr{E}_i) \not\models \mathbb{T}\varphi$. So, there exists an interpretation I such that $I \models \{\mathbb{T}\sigma \mid \sigma \in \Sigma\} \cup \Omega(\mathscr{E}_i)$ but $I \not\models \mathbb{T}\varphi$. There must also exists a conflict-minimal interpretation I' of Σ and $I' <_c I$. Since the assumptions $\overline{\mathbb{C}}\, a : C \in \Omega(\mathscr{E}_i)$ all state that there is no conflict concerning the assertion $a : C$, $I' \models \Omega(\mathscr{E}_i)$ must hold. So, I' is a conflict-minimal interpretation of Σ and $I' \models \Omega(\mathscr{E}_i)$ but according to Lemma 2, $I' \not\models \mathbb{T}\varphi$. This implies $\Sigma \not\models_{<_c} \varphi$. Contradiction.

Hence, every stable extension $\mathscr{E}_i$ contains an argument for φ.

($\Leftarrow$) Let φ be supported by an argument in every stable extension $\mathscr{E}_i$.

Suppose that $\Sigma \not\models_{<_c} \varphi$. Then there is a conflict-minimal interpretation I of Σ and $I \not\models \varphi$. Since I is a conflict-minimal interpretation of Σ, according to Lemma 4, there is a stable extension $\mathscr{E}_i$ and $I \models \Omega(\mathscr{E}_i)$. Since $\mathscr{E}_i$ contains an argument A supporting φ, the assumptions of A must be a subset of $\Omega(\mathscr{E}_i)$, and therefore I satisfies these assumptions. Then, according to Lemma 1, $I \models \varphi$. Contradiction.

Hence, $\Sigma \models_{<_c} \varphi$. □

In Example 5 in the previous section, we saw that one counter-argument implies multiple counter-arguments. The following proposition formalizes this observation.

Proposition 1 *Let* $A_0 = (\{\overline{\mathbb{C}}\alpha_1, \ldots, \overline{\mathbb{C}}\alpha_k\}, \mathbb{C}\alpha_0)$.
Then $A_i = (\{\overline{\mathbb{C}}\alpha_0, \ldots, \overline{\mathbb{C}}\alpha_{i-1}, \overline{\mathbb{C}}\alpha_{i+1}, \ldots, \overline{\mathbb{C}}\alpha_k\}, \mathbb{C}\alpha_i)$ *is an argument for every* $1 \le i \le k$.

Proof The argument A_0 is the result of two tableaux, one for $\mathbb{T}\alpha_0$ and one for $\mathbb{F}\alpha_0$. Then, according to Lemma 1,

$$\{\overline{\mathbb{C}}\alpha_1, \ldots, \overline{\mathbb{C}}\alpha_k\} \cup \{\mathbb{T}\sigma \mid \sigma \in \Sigma\} \models \mathbb{C}\alpha_0$$

where Σ the set of available propositions. This implies that

$$\{\overline{\mathbb{C}}\alpha_0, \ldots, \overline{\mathbb{C}}\alpha_{i-1}, \overline{\mathbb{C}}\alpha_{i+1}, \overline{\mathbb{C}}\alpha_k\} \cup \{\mathbb{T}\sigma \mid \sigma \in \Sigma\} \models \mathbb{C}\alpha_i$$

So, $\{\overline{\mathbb{C}}\alpha_0, \ldots, \overline{\mathbb{C}}\alpha_{i-1}, \overline{\mathbb{C}}\alpha_{i+1}, \overline{\mathbb{C}}\alpha_k\} \cup \{\mathbb{T}\sigma \mid \sigma \in \Sigma\}$ entails both $\mathbb{T}\alpha_i$ and $\mathbb{F}\alpha_i$. Then, according to Lemma 2,

$$A_i = (\{\overline{\mathbb{C}}\alpha_0, \ldots, \overline{\mathbb{C}}\alpha_{i-1}, \overline{\mathbb{C}}\alpha_{i+1}, \ldots, \overline{\mathbb{C}}\alpha_k\}, \mathbb{C}\alpha_i)$$

is an argument for $\mathbb{C}\alpha_i$. □

Related Works

Reasoning in the presences of inconsistent information has been addressed using different approaches. Rescher (29) proposed to focus on maximal consistent subsets of an inconsistent knowledge-base. This proposal was further developed by (8; 15; 25; 30; 31). Brewka and Roos focus on preferred maximal consistent subsets of the knowledge-base while Poole and Huang et al. consider a single consistent subset of the knowledge-base supporting a conclusion. Roos (31) defines a preferential semantics (16; 21; 33) entailing the conclusions that are entailed by every preferred maximal consistent subsets, and provides an assumption-based argumentation system capable of identifying the entailed conclusions.

Paraconsistent logics form another approach to handle inconsistent knowledge bases. Paraconsistent logics have a long history starting with Aristotle. From the beginning of the twentieth century, paraconsistent logics were developed by Orlov (1929), Asenjo (1), da Costa (10), Belnap (3), Priest (26) and others. For a survey of several paraconsistent logics, see for instance (22).

This paper uses the semantics of the paraconsistent logic LP (26; 27) as starting point. Belnap's four-values semantics (3) differs from the LP semantics in allowing the empty set of truth-values. Belnap's semantics was adapted to description logics by Patel-Schneider (23). Ma et al. (20; 18; 19; 17) extend Patel-Schneider's work to more expressive description logics, and propose two new interpretations for the subsumption relation. Qiao and Roos (28) propose another interpretation.

A proof theory based on the semantic tableaux method was first introduced by Beth (4). The semantic tableaux methods have subsequently been developed for many logics. For an overview of several semantic tableaux methods, see (14). Bloesch (5) developed a semantic tableaux method for the paraconsistent logics LP and Belnap's 4-valued logic. This semantic tableaux method has been used as a starting point in this paper.

Argumentation theory has its roots in logic and rhetoric. It dates back to Greek philosophers such as Aristotle. Modern argumentation theory started with the work of Toulmin (35). In Artificial Intelligence, the use of argumentation was promoted by authors such as Pollock (24), Simari and Loui (34), and others. Dung (12) introduced the argumentation framework (AF) in which he abstracts from the structure of the argument and the way the argument is derived. In Dung's argumentation framework, arguments are represented by atoms over which an attack relation is defined. The argumentation framework is used to define an argumentation semantics in terms of sets of conflict-free arguments that defend themselves against attacking arguments. Dung defines three semantics for argumentation frameworks: the grounded, the stable and the preferred semantics. Other authors have proposed additional semantics to overcome some limitations of these three semantics. For an overview, see (3).

This paper uses a special type argumentation system called assumption-based argumentation (ABA). Assumption-based argumentation has been developed since the end of the 1980's (6; 7; 13; 30; 31). Dung et al. (11) formalized assumption-based argumentation in terms of an argumentation framework.

Conclusions

This paper presented a three-valued semantics for $\mathcal{ALC}$, which is based on semantics of the paraconsistent logic LP. An assumption-based argumentation system for identifying conclusions supported by conflict-minimal interpretations was subsequently described. The assumption-based arguments are derived from open branches of a semantic tableaux. The assumptions close open branches by assuming that some proposition will not be assigned the truth-value CONFLICT. No assumptions are needed if conclusions hold is all three-valued interpretations. The described approach has also been implemented.

In future work we intend to extend the approach to the description logic $\mathcal{SROIQ}$. Moreover, we wish to investigate the computational efficiency of our approach in handling inconsistencies.

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