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A preference logic for non-monotonic reasoning

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Abstract

In this paper a generalisation of the first order predicate logic is defined. On the premisses of this logic a preference relation can be specified. By using this preference relation it is possible to describe non-monotonic reasoning in a very natural way. In this non-monotonic logic default rules can be formulated. It is also possible to derive new default rules. Unlike other forms of non-monotonic reasoning, this logic has an executable deduction process. In this deduction process truth maintenance is intergrated.

Contents

1 Introduction

Up till now several forms of non-monotonic reasoning have been suggested. The most important among them are 'Circumscription' [4,5], 'Default logic' [6] and the conditional logic of Delgrande [1,2]. Circumscription one of the oldest forms of non-monotonic logics is developed by McCarthy. Using a second order formula, Circumscription minimalizes the number of objects for which some circumscribed predicate is true. The difficulty with circumscription is that it is not always easy or even possible to collapse the second order formula into a first order formula. We need, however, a first order formula in order to make deductions.

In the default logic of Reiter and the conditional logic of Delgrande default rules are introduced. These rules are introduced as a set of special inference rules in the former logic. To apply such a rule we must be able to determine whether some formulas are derivable. Since the the determination of the derivability of a formula is in general undecidable for a first order logic, we can not make a deduction step unless we use some simplifying assumptions.

In the conditional logic of Delgrande default rules are introduced as a special kind of implication. In this logic it is possible to deduce new default rules by reasoning about default rules using other information. For making default deductions using default rules again we must be able to determine if a formula is derivable.

In this paper a new logic for non-monotonic reasoning is defined. In this logic default rules can be expressed and also new default rules can be deduced. The deduction process of this logic does not depend on an undecidable consistency check or on the collapsing of some second order formula into a first order formula. Instead, a executable deduction process is defined. This deduction process reaches in the limit the theory of a set of premisses. Before this limit is reached wrong believes may be hold. To make it possible to withdraw these wrong believes, a simple truth maintenance system is intergrated in the deduction process.

2 The preference logic

In the preference logic we start with a standard first order predicate logic. We extend this logic in two different ways. Firstly, we allow a partial preference relation to be defined on the set of premisses. Secondly, we change the interpretation of formulas which contain free variables.

The preference relation we introduce is used to resolve contradictions which might be deduced. In cases where a contradiction occurs, the least preferred premiss on which the contradiction is based is removed. To do this, we need to be able to determine these premisses. There are two problems which can arise here. Firstly, there need not to be one least preferred premiss. This means that we do not know which premiss to withdraw. In such cases, we can not solve the inconsistency. It is, however, not necessary that this inconsistency can not be solved at all. Secondly, it is possible that removed premisses have to be placed back because the contradiction causing their removing can not occur any more. This means that the removing of a premiss depends on the other premisses in the subset in which it was a least preferred premiss.

The other extension is the treatment of formulas with free variables. In a standard first order logic a formula $\varphi(\overline{x})$ with free variables $\overline{x}$ which occurs in the set of premisses is equivalent with $\forall\overline{x}\varphi(\overline{x})$. In this logic these formulas are treated differently. The latter formula keeps the same interpretation as the it has in standard first order logic while the former denotes a set of instances of the formula. This new interpretation of the free variables enables us to describe propositions like 'birds can fly'. Due to this new interpretation of premisses containing free variables, we only remove the least preferred instance of the set generated by the premisses after deriving a contradiction.

To be able to solve a contradiction which depends on instances of the premisses, we need to define a preference relation for these instances. Therefore the original preference relation between the premisses is extended for the instances of the premisses. In this extension the preference relation between instances of premisses will be the same as the preference relation between the corresponding premisses.

To determine the premisses which can be believed given the state of knowledge, we introduce a set of justifications. We distinguish two kinds of justifications, in-justifications and out-justifications. The out-justifications are used to determine which premisses have to be withdrawn. The in-justifications are used to determine if a formula can be believed given the set of premisses which are not withdrawn. The in-justifications are also used to determine the set of premisses on which a derived contradiction is based. When this set of premisses has been determined, we can determine the least preferred premiss and can construct an out-justification for it.

In the preference logic we present here a default rule can be represented by a premiss containing free variables. We would like, however, to be able to derive new default rules. To make this possible, we will use justifications containing free variables. If we need an instance φ of a formula ψ containing free variables, this instance will get a justification which is an instance of the justification for the formula ψ. This justification will be the same φ gets when we derive φ from some instances of the premisses.

3 Formal definitions

The preference logic is based on an ordinary first order logic L. A set of premisses Σ of this logic is some subset of this language L. On this set of premisses a partial preference relation can be defined. This partial preference relation for a set of premisses Σ is defined as:

Definition 1 *Let* $(\Sigma, \prec)$ *denote a partial preference relation for a set of premisses* Σ. $(\Sigma, \prec)$ *must be bounded above, combinatorial*[1], *transitive, irreflexive and asymmetric.*

Because premisses containing free variables are viewed as representing a set of instances of those premisses, we introduce an extended set of premisses $\overline{\Sigma}$ which also contains all instances.

Definition 2 $\overline{\Sigma} = \{\varphi(\bar{t}) \mid \varphi(\bar{x}) \in \Sigma$ *and the terms* $\bar{t}$ *of the language* L *are substitutable in* φ *for* $\bar{x}\}$

In case we derive a contradiction we have to withdraw a formula from the extended set of premisses $\overline{\Sigma}$. To be able to do this it is necessary to

[1]An ordering is conbinatorial if and only if every path between two elements in the ordering contains a finite number of elements.

extend the preference relation. This extended preference relation should again be transitive but not reflexive or symmetric. The preference relation for the extended set of premisses is defined as:

Definition 3 *Let* $(\overline{\Sigma}, \prec)$ *denote the preference relation for* $\overline{\Sigma}$.
$(\overline{\Sigma}, \prec)$ *is the smallest partial order containing* $(\Sigma, \prec)$ *and is closed under term substitution in the 'generic' premisses of* Σ.

One should notice that the preference relation $(\overline{\Sigma}, \prec)$ is not always defined as can be seen in the following example.

Example 1
$$\begin{aligned} \Sigma &= \{\varphi(x), \varphi(a), \psi\} \\ (\Sigma, \prec) &= \{\varphi(x) \prec \psi, \psi \prec \varphi(a)\} \end{aligned}$$

Now we have defined the set of extended premisses and their preference relation, we can define the set of possible justifications. We can distinguish two kinds of justifications, 'in-justifications' and 'out-justifications'. The in-justifications are used to denote that a formula is believed if the premisses in the antecedent are believed, while the out-justifications are used to denote that a premiss can no longer be believed (must be withdrawn) if the premisses in the antecedent are believed.

Definition 4
$$\begin{aligned} In\_Just &= \{P \Rightarrow \varphi \mid P \subset \overline{\Sigma} \textit{ and } \varphi \in L\} \\ Out\_Just &= \{P \not\Rightarrow \varphi \mid P \subset \overline{\Sigma} \textit{ and } \varphi \in \overline{\Sigma}\} \end{aligned}$$

4 The reasoning process

The reasoning process of the preference logic exists of finding new justifications for formulas. These justifications are generated by the inference rules. The initial set J_0 contains an in-justification for every formula which is a premiss. These justification indicate that a formula is believed if the corresponding premiss is believed.

Definition 5 $J_0 = \{\{\varphi\} \Rightarrow \varphi \mid \varphi \in \Sigma\}$

Each set of justifications J_i with $i > 0$ is generated from the set J_{i-1} by adding new justifications. How these justifications are determined depends on the deduction system we use. We will continue our description of the preference logic with an axiomatic deduction system for a language L which only contains the logical operators $\rightarrow$ and $\neg$ and the quantor $\forall$. We will use the following axiom scheme

1. Tautologies
2. $\forall x \varphi(x) \rightarrow \varphi(t)$ where the term t is substitutable in φ for x
3. $\forall x(\varphi \rightarrow \psi) \rightarrow (\forall x \varphi \rightarrow \forall x \psi)$
4. $\varphi \rightarrow \forall x \varphi$ where x does not occur in φ

Because we use an axiomatic approach, we have to introduce justifications for the axioms. Since axioms can not be withdrawn an axiom will always have an in-justification with an antecedent equal to the empty set. An instance of the axiom scheme is introduced by the following axiom rule.

Rule 1 *An instance φ of the four axiom scheme gets an in-justification $\emptyset \Rightarrow \varphi$.*

In the deduction system we will use three inference rules, the modus ponens, the generation rule and the contradiction rule. The modus ponens introduces a new in-justification for some formula. This justification is constructed from the justifications for the antecedents of the modus ponens.

Rule 2 *If the formulas φ and $\varphi \rightarrow \psi$ have an in-justification of respectively $P \Rightarrow \varphi$ and $Q \Rightarrow (\varphi \rightarrow \psi)$, then the formula ψ gets an in-justification $(P \cup Q) \Rightarrow \psi$.*

While the modus ponens introduces new a in-justification, the contradiction rule introduces a new out-justification to eliminate a contradiction.

Rule 3 *Let φ and $\neg\varphi$ be formulas with justifications $P \Rightarrow \varphi$ and $Q \Rightarrow \neg\varphi$. If $\{\psi\} = min(P \cup Q)$ —minimal under the preference relation $Pref_{\overline{\Sigma}}$—, then the premiss ψ gets an out-justification $((P \cup Q)/\psi) \not\Rightarrow \psi$.* $(\overline{\Sigma}, <)$

The generation rule is used to generate an instance of a formula φ containing free variables. The justification for this instance is an instance of the justification of the formula φ.

Rule 4 *Let $\varphi(\overline{x})$ be a formula with free variables $\overline{x}$ and let the sequence of term $\overline{t}$ be substitutable in φ for $\overline{x}$.*

- *If $\varphi(\overline{x})$ has an in-justification $P(\overline{x}) \Rightarrow \varphi(\overline{x})$, then the formula $\varphi(\overline{t})$ gets an in-justification $P(\overline{t}) \Rightarrow \varphi(\overline{t})$.*
- *If $\varphi(\overline{x})$ has an out-justification $P(\overline{x}) \not\Rightarrow \varphi(\overline{x})$, then the premiss $\varphi(\overline{t})$ gets an out-justification $P(\overline{t}) \not\Rightarrow \varphi(\overline{t})$.*

We assume that the process which creates a set of justifications J_{k+1} from the set J_k is fair. This means that this process does not forever defer the addition of some possible justification to the set of justifications. If we use a fair process, the following theorems hold.

Theorem 1
For each $i \geq 0$: if $P(\overline{x}) \Rightarrow \varphi(\overline{x}) \in J_i$, then for each sequence of terms $\overline{t}$ substitutable in φ for $\overline{x}$: $P(\overline{t}) \subseteq \overline{\Sigma}$ and $P(\overline{t}) \vdash \varphi(\overline{t})$.

Theorem 2
For each $P \subseteq \overline{\Sigma}$: if $P \vdash \varphi$, then there is some set Q: $Q \subseteq P$ and for some $i \geq 0$: $Q \Rightarrow \varphi \in J_i$.

Theorem 3
For each $i \geq 0$: if $P(\overline{x}) \not\Rightarrow \varphi(\overline{x}) \in J_i$, then for each sequence of terms $\overline{t}$ substitutable in φ for $\overline{x}$: $P(\overline{t}) \cup \{\varphi(\overline{t})\} \subseteq \overline{\Sigma}$, $P(\overline{t}) \cup \{\varphi(\overline{t})\}$ is inconsistent and for each $\psi \in P(\overline{t})$ there holds: $\varphi(\overline{t}) \prec \psi$.

Theorem 4
For each $P \subseteq \overline{\Sigma}$ if P is a minimal inconsistent set and $\{\varphi\} = min(P)$, then for some $i \geq 0$: $P/\varphi \not\Rightarrow \varphi \in J_i$.

Given a set of out-justifications, we want to determine which premisses can be believed. This are of course all premisses which did not have to be withdrawn due to an out-justification. Since the out-justifications which can be applied depend on the set of premisses which are not withdrawn, we get the following fixed point definition.

Definition 6 *The set of premisses Δ_i which may be believed given a set of out-justifications J_i, should satisfy the fixed point equation:*

$$\Delta_i = \overline{\Sigma} - Out_i(\Delta_i)$$

$$\text{where } Out_i(S) = \{\varphi \mid P \not\Rightarrow \varphi \in J_i \text{ and } P \subseteq S\}$$

The set of premisses which satisfy the fixed point definition above have the following important properties.

Property 1 *For each finite i the fixed point Δ_i is unique.*

Property 2 *Let the function $f_i : \overline{\Sigma} \rightarrow \overline{\Sigma}$ be defined as:*

$$f_i(S) = \overline{\Sigma} - Out_i(S)$$

Now for every finite i there exists some finite k such that for each $l > k$ there holds:

$$f_i^k(\overline{\Sigma}) = f_i^l(\overline{\Sigma}) = \Delta_i$$
$$\textit{where } f^k = f(f^{k-1})$$

From these properties it follows that we can construct an algorithm[2] which determines the set of premisses Δ_i

After we have determined the set of premisses which can be believed, we can determine the set of derived formulas which can be believed given the in-justifications. This set is defined as:

Definition 7 *Let J_i be a set of justifications and Δ_i be the corresponding set of premisses. The set of formulas B_i which can be believed is defined as:*

$$B_i = \{\varphi \mid P \Rightarrow \varphi \in J_i \textit{ and } P \subseteq \Delta_i\}$$

Property 3 *For each $\varphi \in B_i : [\Delta_i \vdash \varphi]$*

The set of all justifications which can be derived will be denoted by J_∞. J_∞ is defined as:

Definition 8 $J_\infty = \bigcup_{i \rightarrow \infty} J_i$

The corresponding set of premisses and formulas which can be believed will be denoted by Δ_∞ and by B_∞. For J_∞, Δ_∞ and B_∞ the following properties hold:

[2]In practice it can be impossible to determine $\overline{\Sigma}$. In such cases we can limit ourselves to a subset of $\overline{\Sigma}$ which contains every element of $\overline{\Sigma}$ that occurs in the antecedent of an in- or out-justification or in the consequent of an out-justification.

Property 4 *For each i there holds: $J_i \subseteq J_\infty$*

Property 5

- *Δ_∞ is unique.*
- *$\Delta_\infty \subseteq \overline{\Sigma}$*
- *For each φ: $[\varphi \in (\overline{\Sigma} - \Delta_\infty) \rightarrow \Delta_\infty \cup \{\varphi\}$ is inconsistent$]$*
- *If each minimal inconsistent subset of $\overline{\Sigma}$ has a least preferred element, then Δ_∞ is consistent.*

Property 6 *$Th(\Delta_\infty) = B_\infty$ where $Th(S) = \{\varphi \mid S \vdash \varphi\}$*

5 The semantics for the logic

The semantics we specify in this section is not a traditional one. In fact what is presented here can only be called semantics according to the following definition which can be found in dictionaries: 'the relation between sign and symbols and what they denote or mean'. The reason why we can not use a traditional (Tarskian) semantics is because every premiss of the preference logic is an assumption. This means in principle that every premiss can be withdrawn. One can even assume that $\varphi \wedge \neg\varphi$ holds and prefer this premiss above any other premiss. Still the premiss $\varphi \wedge \neg\varphi$ will be withdrawn independent of the other premisses. Since premisses may contain obscure formulas like $\varphi \wedge \neg\varphi$, it will be clear that a classical semantics will not work here. A premiss which is an assumption can be wrong in the context of other assumed premisses. Therefore the truth value of a sentence can only be defined relative to a consistent set of assumed premisses.

Given a set of premisses there can exist more than one consistent subset. We are not interested in each of these subset but only in those which contain a maximum number of premisses with the highest preference. This can be defined more formally by specifying a preference relation $(P(\overline{\Sigma}), \sqsubset)$ on the sets of premisses using the preference relation between premisses.

Definition 9 *Let $(P(\overline{\Sigma}), \sqsubset)$ denote the preference relation on $P(\overline{\Sigma})$. For each $\Phi, \Psi \subseteq \overline{\Sigma}$ there holds: $\Phi \sqsubseteq \Psi$ if and only if for every $\varphi \in (\Phi - \Psi)$, there is a $\psi \in (\Psi - \Phi)$ such that $\varphi \prec \psi$.*

Given this preference relation for the subsets of premisses, we can prove the following two important theorems. The latter theorem describes which premisses can be believed given the preference relation for the premisses.

Theorem 5 *Let $\mathcal{A}$ denote the set of consistent subsets of $\overline{\Sigma}$*

$$\mathcal{A} = \{\Phi \mid \Phi \subseteq \overline{\Sigma}, \Phi \text{ is consistent}\}$$

Now for every Δ_i there holds: Δ_i is an upper bound of $\mathcal{A}$.

Corollary 1 *Δ_∞ is consistent if and only if $\Delta_\infty = max(\mathcal{A})$*

Theorem 6 *$\Delta_\infty = lub(\mathcal{A})$ where lub stand for the least upper bound.*

6 Some applications

In this section we will try to show the power of the the preference logic by giving some applications. We will first discuss how default rules can be represented in our framework. In the preference logic we can specify formulas which have the same meaning as the normal default rules of Reiter [6]. Defaults like 'Birds can fly' can be specified by generative premisses of the form:

$$\varphi(\overline{x}) \rightarrow \psi(\overline{x})$$

For the default 'Birds can fly' this becomes:

$$Bird(x) \rightarrow Can\_fly(x)$$

By giving the defaults a lower priority than the other premisses, we can assure that an instance of a default is withdrawn if it contradicts with other information.

Example 2

premisses:
1. $Bird(Jan)$
2. $Bird(Piet)$
3. $Penguin(Piet)$
4. $Bird(x) \rightarrow Can\_fly(x)$
5. $\forall x[Penguin(x) \rightarrow \neg Can\_fly(x)]$

preference relations: $1 \succ 4$, $2 \succ 4$, $3 \succ 4$ *and* $5 \succ 4$

conclusions: $Can\_fly(Jan)$
$\neg Can\_fly(Piet)$

In the preference logic new defaults can be deduced. If we add to the example above the premiss 'every eagle is a bird',

$$\forall x[Eagle(x) \rightarrow Bird(x)]$$

then we can deduce the default 'eagles can fly'.

$$Eagle(x) \rightarrow Can\_fly(x)$$

This default can be derived form the premisses 'every eagle is a bird' and 'birds can fly'.

In the deduction of the the default 'eagles can fly' we have deduced a transitive relation between a default and an implication. The possibility to derive such a transitive relation is not always wanted. To avoid unwanted transitive relations among defaults and other implications, Reiter and Criscuolo [7] argued that it is not enough to have only normal defaults. They argue that semi normal defaults are required. A semi normal default is used to describe a default with exceptions on the application of this default. This makes it possible to avoid unwanted transitive relations. In the preference logic we do not have something equivalent to a semi normal default. In this logic we avoid these unwanted transitive relations by adding new defaults and by defining the right preference relation.

Example 3

- *University students are normally adults.*
- *Adults are normally employed.*

From these two sentences we can conclude that university students are normally employed. We know, however, that university students are normally unemployed. By adding this information with the correct preference relation, we can avoid the unwanted transitive relation.

premisses:
1. $Univ\_Stud(x) \rightarrow Adult(x)$
2. $Adult(x) \rightarrow Employed(x)$
3. $Univ\_Stud(x) \rightarrow \neg Employed(x)$

preference relations: $1 \succ 2$ *and* $3 \succ 2$

If we do not want to conclude either that university students are normally employed or that university students are normally unemployed, then we have to use a modal logic. In a modal logic we can describe an implication more naturally. With a sentence "if A then B" we actually mean: 'in all situation where A holds, B should also hold'. In a modal logic we can describe this with:

$$\Box(\neg A \vee B)$$

So if we do not want to conclude either that university students are normally employed or that university students are normally unemployed, and we are using a modal logic in which $\varphi \rightarrow \psi$ *is equivalent to* $\Box(\neg\varphi \vee \psi)$*, then we should replace the third premiss by:*

$$\neg(Univ\_Stud(x) \rightarrow Employed(x))$$

This premiss says that the implication does not hold.

Hanks and McDermott [3] identified a temporal projection problem for which they showed that the non-monotonic logic they considered are to week to model it. They specified there problem in a situation calculus which we have reformulated in the preference logic.

1. $\forall s[T(Loaded, Result(Load, s))]$
2. $\forall s[T(Loaded, s) \rightarrow T(Dead, Result(Shoot, s))]$
3. $\forall s[\neg(T(Alive, s) \wedge T(Dead, s))]$
4. $T(f, s) \rightarrow T(f, Result(e, s))$
5. $T(Alive, S_0)$
6. $S_1 = Result(Load, S_0)$
7. $S_2 = Result(Wait, S_1)$
8. $S_3 = Result(Shoot, S_2)$

$1 \succ 4,\ 2 \succ 4,\ 3 \succ 4,\ 5 \succ 4,\ 6 \succ 4,\ 7 \succ 4$ en $8 \succ 4$.

From these premisses of the problem we can derive $T(Dead, S_3)$ and $T(Alive, S_3)$ causing a contradiction. Because in both the deduction of $T(Alive, S_3)$ and $T(Dead, S_3)$ an instance of the same default 4 is used, we do not know which instance we have to prefer. About the same problem arise when we use some other form of non-monotonic reasoning. The solution to this problem which was suggested by Hanks and McDermott exists in allowing only those necessary exception on a default which occur the latest in time. We can model this in the preference logic if we allow a preference relation to be specified between instances of a premiss. We can solve the shooting problem by extending the preference relation:

Definition 10 *The new preference relation is the transitive closure of:*

- $(\overline{\Sigma}, \prec)$
- *Let* $\varphi(f,e,s)$ *denote* $T(f,s) \rightarrow T(f, Result(e,s))$. *For each pair of instances* $\varphi(f,e,S)$ *and* $\varphi(f,e,Result(e,S))$*:*
  $\varphi(f,e,S) \succ \varphi(f,e,Result(e,S))$.

An area where the preference logic can be used in the formalization of inheritance hierarchies with exceptions. To formalize inheritance networks we need to be able to specify that relations which hold between objects of some class can be overruled by relations which hold between objects of a more specific class. This is for example more or less identical to the inferential distance algorithm of Touretzky [8]. We can formalize the inheritance networks with exceptions in the preference logic by defining the following preference relations between premisses.

Definition 11 *For each premiss* $\varphi(\overline{x}) \rightarrow \chi(\overline{x})$ *and for each premiss* $\psi(\overline{x}) \rightarrow \omega(\overline{x})$ *if* $\forall \overline{x}(\varphi(\overline{x}) \rightarrow \psi(\overline{x}))$, *then* $[\varphi(\overline{x}) \rightarrow \chi(\overline{x})] \succ [\psi(\overline{x}) \rightarrow \omega(\overline{x})]$

Using this preference relation, we can also handle relations which hold between two diffent inheritance hierarchies.

Example 4

premisses:
1. $\forall x[Royal\_Elephant(x) \rightarrow Elephant(x)]$
2. $Elephant(x) \wedge Mouse(y) \rightarrow \neg Like(x,y)$
3. $Royal\_Elephant(x) \wedge Mouse(y) \wedge White(y) \rightarrow Like(x,y)$

preference relation: $2 \prec 3$

7 Conclusion

In the preceding sections we have presented a generalisation of the predicate logic. This generalisation makes it possible to model default reasoning in a very natural way. The main advantage of this logic is its executable deduction process which approaches in the limit the theory of the set of preferred premisses.

In the applications we have seen how default rules can be modeled. Also more complex forms of default reasoning such as inheritance networks can be modeled in the preference logic. Further we have seen how the shooting problem of Hanks and McDermott can be solved using the preference logic. The solution we have presented makes it possible to apply defaults in a situation calculus. Other possible application for the preference logic like counterfactual reasoning are still unexplored.

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References

[1] Delgrande, J. P., First-order conditional logic for prototypical properties, *Artificial Intelligence* **33** (1987) 105-130.

[2] Delgrande, J. P., An approach to default reasoning based on first-order conditional logic: revised report, *Artificial Intelligence* **36** (1988) 63-90.

[3] Hanks, S., McDermott, D., Nonmonotonic logic and temporal projection, *Artificial Intelligence* **33** (1987) 379-412.

[4] McCarthy, J., Circumscription - a form of non-monotonic reasoning, *Artificial Intelligence* **13** (1980) 27-39.

[5] McCarthy, J., Applications of circumscription, *Artificial Intelligence* **28** (1986) 89-116.

[6] Reiter, R., A logic for default reasoning, *Artificial Intelligence* **13** (1980) 81-132.

[7] Reiter, R., Criscuolo, G., On interacting defaults, *Proceedings of the 7th IJCAI* (1981) 270-276.

[8] Touretzky, D. S., *The mathematics of inheritance systems*, Pitman, London, Morgan Kaufman Publishers Inc., Los Altos California (1986).

Appendix

Theorem 1

For each $i \geq 0$: if $P(\overline{x}) \Rightarrow \varphi(\overline{x}) \in J_i$, then for each sequence of terms $\overline{t}$: $P(\overline{t}) \subseteq \overline{\Sigma}$ and $P(\overline{t}) \vdash \varphi(\overline{t})$.

Proof

We will prove the theorem by induction on the index of the sets of justifications.

Initialisation step For $i = 0$: $\{\varphi(\overline{x})\} \Rightarrow \varphi(\overline{x}) \in J_0$ if and only if $\varphi(\overline{x}) \in \Sigma$. By definition 2 we have for each sequence of terms $\overline{t}$: $\varphi(\overline{t}) \in \overline{\Sigma}$. Therefore for each sequence of terms $\overline{t}$: $\{\varphi(\overline{t})\} \vdash \varphi(\overline{t})$.

Induction hypothesis For each $i \leq k$ if $P(\overline{x}) \Rightarrow \varphi(\overline{x}) \in J_i$, then for each sequence of terms $\overline{t}$: $P(\overline{t}) \subseteq \overline{\Sigma}$ and $P(\overline{t}) \vdash \varphi(\overline{t})$.

Induction step Suppose that $P(\overline{x}) \Rightarrow \varphi(\overline{x}) \in J_{k+1}$. $P(\overline{x}) \Rightarrow \varphi(\overline{x}) \in J_{k+1}$ if and only if $P(\overline{x}) \Rightarrow \varphi(\overline{x}) \in J_k$ or $P(\overline{x}) \Rightarrow \varphi(\overline{x})$ has been added by rule 1, 2 or 4.

If $P(\overline{x}) \Rightarrow \varphi(\overline{x}) \in J_k$, then by the induction hypothesis we have for each sequence of terms $\overline{t}$: $P(\overline{t}) \subseteq \overline{\Sigma}$ and $P(\overline{t}) \vdash \varphi(\overline{t})$.

If $P(\overline{x}) \Rightarrow \varphi(\overline{x})$ is introduced by rule 1, then it is an axiom. Therefore $P(\overline{x}) = \emptyset$ and for each sequence of terms $\overline{t}$: $\vdash \varphi(\overline{t})$.

If $P(\overline{x}) \Rightarrow \varphi(\overline{x})$ is introduced by rule 2, then there is a $Q(\overline{x}) \Rightarrow \alpha(\overline{x}) \in J_k$, $R(\overline{x}) \Rightarrow (\alpha \rightarrow \varphi)(\overline{x}) \in J_k$ and $P(\overline{x}) = Q(\overline{x}) \cup R(\overline{x})$. According to the induction hypothesis for each sequence of terms $\overline{t}$: $Q(\overline{t}), R(\overline{t}) \subseteq \overline{\Sigma}$, $Q(\overline{t}) \vdash \alpha(\overline{t})$ and $R(\overline{t}) \vdash (\alpha \rightarrow \varphi)(\overline{t})$. Therefore for each sequence of terms $\overline{t}$: $P(\overline{t}) \subseteq \overline{\Sigma}$ and $P(\overline{t}) \vdash \varphi(\overline{t})$.

If $P(\overline{x}) \Rightarrow \varphi(\overline{x})$ is introduced by rule 4, then there is a $P'(\overline{x}) \Rightarrow \varphi'(\overline{x}) \in J_k$ and for some sequence of terms $\overline{t}$: $P(\overline{x}) = P'(\overline{t})$ and $\varphi(\overline{x}) = \varphi'(t)$. Now by the induction hypothesis there holds for each sequence of terms $\overline{t}$: $P'(\overline{t}) \subseteq \overline{\Sigma}$ and $P'(\overline{t}) \vdash \varphi'(\overline{t})$. Hence for each sequence of terms $\overline{t}$: $P(\overline{t}) \subseteq \overline{\Sigma}$ and $P(\overline{t}) \vdash \varphi(\overline{t})$.

Theorem 2

For each $P \subseteq \overline{\Sigma}$: if $P \vdash \varphi$, then there is some set Q: $Q \subseteq P$ and for some $i \geq 0$: $Q \Rightarrow \varphi \in J_i$.

Proof Let $P \subseteq \overline{\Sigma}$ and $P \vdash \varphi$. Since $P \vdash \varphi$ there exists a deduction sequence $\langle \varphi_0, \varphi_1, ..., \varphi_n \rangle$ such that $\varphi_n = \varphi$ and for each $j \leq n$: either

- $\varphi_j \in P$, or
- φ_j is an axiom, or
- there exists a φ_k and a φ_l with $k, l < j$ and $\varphi_l = \varphi_k \rightarrow \varphi_j$.

Now we will proof the theorem using induction to the length of the deduction sequence.

Initialisation step Let $\langle \varphi_0 \rangle$ be a deduction sequence for $P \vdash \varphi$.

If $\varphi_0 \in P$, then we have two possibilities. If $\varphi_0 \in \Sigma$, then $\{\varphi_0\} \Rightarrow \varphi_0 \in J_0$. If $\varphi_0 \notin \Sigma$ but $\varphi_0 \in \overline{\Sigma}$, then there exists a $\psi(\overline{x}) \in \Sigma$ and for some sequence of terms: $\psi(\overline{x}) = \varphi$. Since $\{\psi(\overline{x})\} \Rightarrow \psi(\overline{x}) \in J_0$, there exists an i_0 such that: $\{\psi(\overline{t})\} \Rightarrow \psi(\overline{t}) \in J_{i_0}$ and $\{\psi(\overline{t})\} \Rightarrow \psi(\overline{t})$ is added by rule 4a.

If φ_0 is an axiom, then there exists some $i_0 \geq 0$ such that $J_{i_0} = J_{i_0 - 1} \cup \emptyset \Rightarrow \varphi_0$ and $\emptyset \Rightarrow \varphi_0$ is added by rule 1.

Hence the theorem holds for a deduction sequence of lenght 1.

Induction hypothesis For each deduction deduction $P \vdash \varphi_n$ with a deduction sequence $\langle \varphi_0, \varphi_1, ..., \varphi_n \rangle$ and $n \leq m$ there exists some i_n such that $Q \Rightarrow \varphi_n \in J_{i_n}$ and $Q \subset P$.

Induction step Let $\langle \varphi_0, \varphi_1, ..., \varphi_{m+1} \rangle$ be a deduction sequence for $P \vdash \varphi_{m+1}$

If $\varphi_{m+1} \in P$, then $\{\varphi_{m+1}\} \Rightarrow \varphi_{m+1} \in J_0$.

If φ_{m+1} is an axiom, then there is some i_{m+1} such that $J_{i_{m+1}} = J_{i_{m+1}-1} \cup \emptyset \Rightarrow \varphi_{m+1}$ and $\emptyset \Rightarrow \varphi_{m+1}$ is added by rule 1.

If there exists a φ_k and a φ_l with $k, l \leq m+1$ and $\varphi_l = \varphi_k \rightarrow \varphi_{m+1}$, then by the induction hypothesis there exists some i_k and some i_l such that $Q \Rightarrow \varphi_k \in J_{i_k}$, $R \Rightarrow (\varphi_k \rightarrow \varphi_{m+1}) \in J_{i_l}$ and $Q, R \subseteq P$. Now there exists some i_{m+1} with $i_k, i_l < i_{m+1}$ such that $S \Rightarrow \varphi_{m+1} \in J_{i_{m+1}}$ and $S = Q \cup R$.

Hence there exists some i_{m+1} and some $S \subseteq P$ such that $S \Rightarrow \varphi_{m+1} \in J_{i_{m+1}}$.

Theorem 3
For each $i \geq 0$: if $P(\overline{x}) \not\Rightarrow \varphi(\overline{x}) \in J_i$, then for each sequence of terms $\overline{t}$: $P(\overline{t}) \cup \{\varphi(\overline{t})\} \subseteq \overline{\Sigma}$, $P(\overline{t}) \cup \{\varphi(\overline{t})\}$ is inconsistent and for each $\psi \in P(\overline{t})$ there holds: $\varphi(\overline{t}) \prec \psi$.

Proof We will proof this theorem using induction to the index of the set of justifications.

Initialisation step For $i = 0$: the theorem holds vacuously because there is no $P(\overline{x}) \not\Rightarrow \varphi(\overline{x}) \in J_0$.

Induction hypothesis For each $i \leq k$: if $P(\overline{x}) \not\Rightarrow \varphi(\overline{x}) \in J_i$, then for each sequence of terms $\overline{t}$: $P(\overline{t}) \cup \{\varphi(\overline{t})\} \subseteq \overline{\Sigma}$, $P(\overline{t}) \cup \{\varphi(\overline{t})\}$ is inconsistent and for each $\psi \in P(\overline{t})$ there holds: $\varphi(\overline{t}) \prec \psi$.

Induction step Suppose that $P(\overline{x}) \not\Rightarrow \varphi(\overline{x}) \in J_{k+1}$. $P(\overline{x}) \not\Rightarrow \varphi(\overline{x}) \in J_{k+1}$ if and only if $P(\overline{x}) \not\Rightarrow \varphi(\overline{x}) \in J_k$ or $P(\overline{x}) \not\Rightarrow \varphi(\overline{x})$ has been added by rule 3 or 4.

If $P(\overline{x}) \not\Rightarrow \varphi(\overline{x}) \in J_k$, then by the induction hypothesis we have for each sequence of terms $\overline{t}$: $P(\overline{t}) \cup \{\varphi(\overline{t})\} \subseteq \overline{\Sigma}$, $P(\overline{t}) \cup \{\varphi(\overline{t})\}$ is inconsistent and for each $\psi \in P(\overline{t})$ there holds: $\varphi(\overline{t}) \prec \psi$.

If $P(\overline{x}) \not\Rightarrow \varphi(\overline{x})$ is introduced by rule 3, then there is a $R(\overline{x}) \Rightarrow \alpha(\overline{x}) \in J_k$, $Q(\overline{x}) \Rightarrow \neg\alpha(\overline{x}) \in J_k$, $P(\overline{x}) = (R(\overline{x}) \cup Q(\overline{x}))/\varphi(\overline{x})$ and $\{\varphi(\overline{x})\} = min(R(\overline{x}) \cup Q(\overline{x}))$. By theorem 1 we have for each sequence of terms $\overline{t}$: $R(\overline{t}), Q(\overline{t}) \subseteq \overline{\Sigma}$, $R(\overline{t}) \vdash \alpha(\overline{t})$ and $Q(\overline{t}) \vdash \neg\alpha(\overline{t})$. Hence for each sequence of terms $\overline{t}$: $P(\overline{t}) \cup \{\varphi(\overline{t})\} \subseteq \overline{\Sigma}$, $P(\overline{t}) \cup \{\varphi(\overline{t})\}$ is inconsistent and for each $\psi \in P(\overline{t})$: $\varphi(\overline{t}) \prec \psi$.

If $P(\overline{x}) \not\Rightarrow \varphi(\overline{x})$ is introduced by rule 4, then there is a $P'(\overline{x}) \not\Rightarrow \varphi'(\overline{x}) \in J_k$ such that for some sequence of terms $\overline{t}$: $P'(\overline{t}) = P(\overline{x})$ and $\varphi'(\overline{t}) = \varphi(\overline{x})$. By the induction hypothesis there holds for each sequence of terms $\overline{t}$: $P'(\overline{t}) \cup \{\varphi'(\overline{t})\} \subseteq \overline{\Sigma}$, $P'(\overline{t}) \cup \{\varphi'(\overline{t})\}$ is inconsistent and for each $\psi \in P'(\overline{t})$ there holds: $\varphi'(\overline{t}) \prec \psi$. Hence for each sequence of terms $\overline{t}$: $P(\overline{t}) \cup \{\varphi(\overline{t})\} \subseteq \overline{\Sigma}$, $P(\overline{t}) \cup \{\varphi(\overline{t})\}$ is inconsistent and for each $\psi \in P(\overline{t})$ there holds: $\varphi(\overline{t}) \prec \psi$.

Theorem 4
For each $P \subseteq \overline{\Sigma}$ if P is a minimal inconsistent set and $\{\varphi\} = min(P)$, then for some $i \geq 0$: $P/\varphi \not\Rightarrow \varphi \in J_i$.

Proof Let P be a minimal inconsistent subset of $\overline{\Sigma}$ with $\{\varphi\} = min(P)$. Since P is inconsistent there exists a formula α such that $P \vdash \alpha$ and $P \vdash \neg\alpha$. By theorem 2 there exists a $Q \subseteq P$ and for some $j \geq 0$: $Q \Rightarrow \alpha \in J_j$ and also an $R \subseteq P$ and for some $k \geq 0$: $R \Rightarrow \neg\alpha \in J_k$. Since P in a minimal inconsistent subset of $\overline{\Sigma}$, $P = Q \cup R$. Now there exists a $l > j, k$ such that $J_{l+1} = J_l \cup P/\varphi \not\Rightarrow \varphi$ and $P/\varphi \not\Rightarrow \varphi$ is added by rule 3.

Hence for some $i \geq 0$: $P/\varphi \not\Rightarrow \varphi \in J_i$.

Lemma 1 *If there exists two fixed points X and Y for a set of justifications J, then for each φ: $\varphi \in X$ and $\varphi \notin Y$ there exists a ψ: $\psi \in X$ and $\psi \notin Y$ and $\varphi \prec \psi$.*

Proof Let $\varphi \in X$ and $\varphi \notin Y$. Then there exists a $P \not\Rightarrow \varphi \in J$ and $P \subseteq Y$. Since $\varphi \in X$ there exists an $\eta \in P$, $\eta \notin X$ and $\varphi \prec \eta$. Then there exists a $Q \not\Rightarrow \eta \in J$ and $Q \subseteq X$. Since $\eta \in Y$ there exists a $\psi \in Q$, $\psi \notin Y$ and $\eta \prec \psi$. Hence for each $\varphi \in X$ and $\varphi \notin Y$ there exists a $\psi \in X$, $\psi \notin Y$ and $\varphi \prec \psi$.

Property 1 *For each finite i the fixed point Δ_i is unique.*

Proof Suppose that Δ_i is not unique. Then there exists at least one subset Δ'_i of $\overline{\Sigma}$ such that for some $\varphi \in \Delta_i$: $\varphi \notin \Delta'_i$. Now let O denote the set of all formulas φ such that $\varphi \in \Delta_i$ and $\varphi \notin \Delta'_i$. For every formula φ of O there exists an out-justification $P \not\Rightarrow \varphi$ in J_i. Since i is finite, so is the number of out-justifications in J_i and therefore $|O|$ is also finite. Because there exists at least two fixed points and because $|O|$ is finite, there exists a $\varphi \in max(O)$. Now by lemma 1 there exists a $\psi \in \Delta_i$, $\psi \notin \Delta'_i$ and $\varphi \prec \psi$. Therefore $\psi \in O$. But then $\varphi \notin max(O)$, contradiction.

Hence Δ_i is unique.

Lemma 2 *For every $k \geq 0$: $f_i^{2k+2}(\overline{\Sigma}) \subseteq f_i^{2k}(\overline{\Sigma})$.*

Proof We will prove that for every $k \geq 0$: $f_i^{2k+2}(\overline{\Sigma}) \subseteq f_i^{2k}(\overline{\Sigma})$ using induction on the index k.

Initialisation step $f_i^2(\overline{\Sigma}) = \overline{\Sigma} - Out_i(f_i(\overline{\Sigma}))$. Hence $f_i^2(\overline{\Sigma}) \subseteq \overline{\Sigma}$.

Induction hypothesis For every $j \leq k$: $f_i^{2j+2}(\overline{\Sigma}) \subseteq f_i^{2j}(\overline{\Sigma})$.

Induction step Suppose that $f_i^{2(k+1)+2}(\overline{\Sigma}) \not\subseteq f_i^{2(k+1)}(\overline{\Sigma})$. Then for some $\varphi \in \overline{\Sigma}$: $\varphi \in f_i^{2k+4}(\overline{\Sigma})$ and $\varphi \notin f_i^{2k+2}(\overline{\Sigma})$. So there exists a $P \not\Rightarrow \varphi$, $P \not\subseteq f_i^{2k+3}(\overline{\Sigma})$ and $P \subseteq f_i^{2k+1}(\overline{\Sigma})$. Since $P \not\subseteq f_i^{2k+3}(\overline{\Sigma})$ there exists a $\psi \in P$, $\psi \notin f_i^{2k+3}(\overline{\Sigma})$ and $\psi \in f_i^{2k+1}(\overline{\Sigma})$. Because $\psi \notin f_i^{2k+3}(\overline{\Sigma})$ there exists a $Q \not\Rightarrow \psi \in J_i$, $Q \subseteq f_i^{2k+2}(\overline{\Sigma})$ and $Q \not\subseteq f_i^{2k}(\overline{\Sigma})$. Therefore there exists an $\eta \in Q$, $\eta \notin f_i^{2k}(\overline{\Sigma})$ and $\eta \in f_i^{2k+2}(\overline{\Sigma})$. Hence $f_i^{2k+2}(\overline{\Sigma}) \not\subseteq f_i^{2k}(\overline{\Sigma})$. Since by the induction hypothesis $f_i^{2k+2}(\overline{\Sigma}) \subseteq f_i^{2k}(\overline{\Sigma})$, we have a contradiction.

Hence $f_i^{2(k+1)+2}(\overline{\Sigma}) \subseteq f_i^{2(k+1)}(\overline{\Sigma})$.

Lemma 3 *For every $k \geq 0$: $\Delta_i \subseteq f_i^{2k}(\overline{\Sigma}) \subseteq \overline{\Sigma}$.*

Proof By the definition of the function f we have for each $k \geq 0$: $f_i^{2k}(\overline{\Sigma}) \subseteq \overline{\Sigma}$. We will prove for every $k \geq 0$: $\Delta_i \subseteq f_i^{2k}(\overline{\Sigma})$ by induction to the index k.

Initialisation step $k = 0$: $\Delta_i \subseteq \overline{\Sigma} = f_i^{2k}(\overline{\Sigma})$.

Induction hypothesis For every $j \leq k$: $\Delta_i \subseteq f_i^{2j}(\overline{\Sigma})$.

Induction step Suppose $\Delta_i \not\subseteq f_i^{2k+2}(\overline{\Sigma})$. Then there exists a $\varphi \in \Delta_i$ and $\varphi \notin f_i^{2k+2}(\overline{\Sigma})$. Since $\varphi \notin f_i^{2k+2}(\overline{\Sigma})$, there exists a $P \not\Rightarrow \varphi \in J_i$ and $P \subseteq f_i^{2k+1}(\overline{\Sigma})$. Because $\varphi \in \Delta_i$, there exists a $\psi \in P$, $\psi \in f_i^{2k+1}(\overline{\Sigma})$ and $\psi \notin \Delta_i$. Hence there exists a $Q \not\Rightarrow \psi \in J_i$ and $Q \subseteq \Delta_i$. Since $\psi \in f_i^{2k+1}(\overline{\Sigma})$, there exists an $\eta \in Q$, $\eta \in \Delta_i$ and $\eta \notin f_i^{2k}(\overline{\Sigma})$. Hence $\Delta_i \not\subseteq f_i^{2k}(\overline{\Sigma})$. By the induction hypothesis we have $\Delta_i \subseteq f_i^{2k}(\overline{\Sigma})$. Contradiction.

Lemma 4 *For every $X \subseteq \overline{\Sigma}$: X is a fixed point of $f \circ f$ if and only if X is a fixed point of f.*

Proof The 'if' part of this lemma is trivial. We only have to proof the 'only if' part.

Suppose that X is a fixed point of $f \circ f$ but not of f. Then $f(X) = Y$, $f(Y) = X$ and $X \neq Y$. Therefore $X \not\subseteq Y$ or $Y \not\subseteq X$.

Suppose that $X \not\subseteq Y$. Let $D = X - Y$. Because $X \not\subseteq Y$ and i is finite there exists a $\varphi \in max(D)$. Since $\varphi \in D$ and $\varphi \notin Y$, there exists a $P \not\Rightarrow \varphi \in J_i$ such that $P \subseteq X$ and $P \not\subseteq Y$. Therefore there exists a $\psi \in P$, $\psi \in X$, $\psi \notin Y$ and $\varphi \prec \psi$. Hence $\psi \in D$. But then $\varphi \notin max(D)$, contradiction. Hence $X \subseteq Y$.

For $Y \subseteq X$ the proof is the same.

Hence if X is a fixed point of $f \circ f$, then it is a fixed point of f.

Property 2 *Let the function $f_i : \overline{\Sigma} \to \overline{\Sigma}$ be defined as:*

$$f_i(S) = \overline{\Sigma} - Out_i(S)$$

Now for every finite i there exists some finite k such that for each $l > k$ there holds:

$$f_i^k(\overline{\Sigma}) = f_i^l(\overline{\Sigma}) = \Delta_i$$

where

$$f^k = f(f^{k-1})$$

Proof According to lemma 2,3 we have for every $k \geq 0$: $\Delta_i \subseteq f_i^{2k+2}(\overline{\Sigma}) \subseteq f_i^{2k}(\overline{\Sigma}) \subseteq \overline{\Sigma}$. Since i is finite, so is $|\overline{\Sigma} - \Delta_i|$. Therefore for some finite k: $f_i^{2k+2}(\overline{\Sigma}) = f_i^{2k}(\overline{\Sigma})$. So $f_i^{2k}(\overline{\Sigma})$ is a fixed point of $f \circ f$. Therefore by lemma 4: $f_i^{2k}(\overline{\Sigma})$ is a fixed point of f. Since Δ_i is a fixed point of f and Δ_i is unique, $f_i^{2k}(\overline{\Sigma}) = \Delta_i$ for some finite k.

Property 3 *For each $\varphi \in B_i : [\Delta_i \vdash \varphi]$*

Proof Suppose $\varphi \in B_i$. Then there exists a $P \Rightarrow \varphi \in J_i$ and $P \subseteq \Delta_i$. Therefore by theorem 1: $P \vdash \varphi$. Because $P \subseteq \Delta_i$, $\Delta_i \vdash \varphi$.

Property 4 *For each i there holds: $J_i \subseteq J_\infty$*

Proof This property follows immediate from the definition of J_∞.

Property 5

- *Δ_∞ is unique.*

- $\Delta_\infty \subseteq \overline{\Sigma}$
- *For each* φ*:* $[\varphi \in (\overline{\Sigma} - \Delta_\infty) \rightarrow \Delta_\infty \cup \{\varphi\}$ *is inconsistent*$]$
- *If each minimal inconsistent subset of* $\overline{\Sigma}$ *has a least preferred element, then* Δ_∞ *is consistent.*

Proof

- Suppose that Δ_∞ is not unique. Then there exists at least one subset Δ'_∞ of $\overline{\Sigma}$ such that for some $\varphi \in \Delta_\infty$: $\varphi \notin \Delta'_\infty$. Now let O denote the set of all formulas φ such that $\varphi \in \Delta_\infty$ and $\varphi \notin \Delta'_\infty$. For every formula φ of O there exists an out-justification $P \not\Rightarrow \varphi$ in J_∞. Since $(\overline{\Sigma}, \prec)$ is bounded and combinatorial, there exists a $\varphi \in max(O)$. Now by lemma 1 there exists a $\psi \in \Delta_\infty$, $\psi \notin \Delta'_\infty$ and $\varphi \prec \psi$. Therefore $\psi \in O$. But then $\varphi \notin max(O)$, contradiction.

  Hence Δ_∞ is unique.

- Since $\Delta_\infty = \overline{\Sigma} - Out_\infty(\Delta_\infty)$, we have $\Delta_\infty \subseteq \overline{\Sigma}$.

- If $\varphi \in (\overline{\Sigma} - \Delta_\infty)$, then $\varphi \in Out_\infty(\Delta_\infty)$. Therefore there exists a $P \not\Rightarrow \varphi \in J_\infty$ and $P \subseteq \Delta_\infty$. Since $P \not\Rightarrow \varphi \in J_\infty$, $P \cup \{\varphi\}$ is inconsistent.

  Hence $\Delta_\infty \cup \{\varphi\}$ is inconsistent.

- Let each minimal inconsistent subset of $\overline{\Sigma}$ have a least preferred element. Suppose that Δ_∞ is inconsistent. Then there exists a minimal inconsistent subset M of Δ_∞. Since $\Delta_\infty \subseteq \overline{\Sigma}$, M has a least preferred element φ. Therefore by theorem 4 there exists an i with $M/\varphi \not\Rightarrow \varphi \in J_i$. Hence $M/\varphi \not\Rightarrow \varphi \in J_\infty$. Because $M/\varphi \subseteq \Delta_\infty$, $\varphi \notin \Delta_\infty$. Contradiction.

Property 6 $Th(\Delta_\infty) = B_\infty$

where $Th(S) = \{\varphi \mid S \vdash \varphi\}$

Proof $B_\infty \subseteq Th(\Delta_\infty)$ because by property 4: if $\varphi \in B_\infty$, then $\Delta_\infty \vdash \varphi$.

Suppose $B_\infty \subset Th(\Delta_\infty)$. Then there exists a φ: $\varphi \notin B_\infty$ and $\Delta_\infty \vdash \varphi$. By theorem 2 there exists an i such that $P \Rightarrow \varphi \in J_i$ and $P \subseteq \Delta_\infty$. Therefore $P \Rightarrow \varphi \in J_\infty$, and $P \subseteq \Delta_\infty$. Hence $\varphi \in B_\infty$, contradiction.

Hence $B_\infty = Th(\Delta_\infty)$.

Theorem 5 *Let $\mathcal{A}$ denote the set of consistent subsets of $\overline{\Sigma}$*

$$\mathcal{A} = \{\Phi \mid \Phi \subseteq \overline{\Sigma}, \Phi \textit{ is consistent}\}$$

Now for every Δ_i there holds: Δ_i is an upper bound of $\mathcal{A}$.

Proof Let $X \in \mathcal{A}$ and $\varphi \in (X - \Delta_\infty)$. Since $\varphi \notin \Delta_\infty$, there exists a $P \not\Rightarrow \varphi \in J_\infty$ and $P \subseteq \Delta_\infty$. Suppose $P \subseteq X$. Then since $\varphi \in X$, X is inconsistent and $X \notin \mathcal{A}$. Contradiction. Hence $P \not\subseteq X$. Therefore there exists a $\psi \in P$ and $\psi \notin X$. Hence $\psi \in (\Delta_\infty - X)$ and $\varphi \prec \psi$.

Hence for every $X \in \mathcal{A}$: $X \sqsubseteq \Delta_\infty$. Therefore Δ_∞ is an upper bound of $\mathcal{A}$.

Corollary 1 *Δ_∞ is consistent if and only if $\Delta_\infty = max(\mathcal{A})$*

Proof Since Δ_∞ is consistent and Δ_∞ is an upper bound, $\Delta_\infty = max(\mathcal{A})$. Since $\Delta_\infty = max(\mathcal{A})$, $\Delta_\infty \in \mathcal{A}$. Hence Δ_∞ is consistent.

Lemma 5 *Let I denote an inconsistent set of formulas with $\{\varphi\} = min(I)$. Then there exists a $P \not\Rightarrow \varphi \in J_\infty$ and $P \subseteq I/\varphi$.*

Proof Since I is inconsistent there exists an η such that: $I \vdash \eta$ and $I \vdash \neg\eta$. Therefore there exists an i such that for some $Q, R \subseteq I$: $Q \Rightarrow \eta \in J_i$ and $R \Rightarrow \eta \in J_i$. Since $\varphi \to (\eta \to (\neg(\eta \to \neg\varphi)))$ and $\neg\eta \to (\eta \to \neg\varphi)$ are axioms, there exists a $j > i$ such that: $Q \cup \{\varphi\} \Rightarrow \neg(\eta \to \neg\varphi) \in J_j$ and $R \Rightarrow \eta \to \neg\varphi \in J_j$. Since $\{\varphi\} = min(Q \cup R \cup \{\varphi\})$, by rule 3 there exists a $k > j$ such that $(Q \cup R)/\varphi \not\Rightarrow \varphi \in J_k$.

Hence $P \not\Rightarrow \varphi \in J_\infty$ with $P = (Q \cup R)/\varphi$.

Theorem 6 $\Delta_\infty = lub(\mathcal{A})$

where lub stand for the least upper bound.

Proof Suppose $\Delta_\infty \neq lub(\mathcal{A})$. Then there exists a X with $lub(\mathcal{A}) \sqsubseteq X \sqsubset \Delta_\infty$. Because $X \sqsubset \Delta_\infty$, $X \subset \Delta_\infty$ or $|X - \Delta_\infty| \geq 1$. Let $D(\alpha) = \{\beta \mid \beta \in X \text{ and } \alpha \prec \beta\}$.

Suppose $X \subset \Delta_\infty$. Since $X \subset \Delta_\infty$, there exists a $\varphi \in \Delta_\infty$ and $\varphi \notin X$. Assume that $D(\varphi) \cup \{\varphi\}$ is consistent. Then $D(\varphi) \cup \{\varphi\} \in \mathcal{A}$. Since

$(D(\varphi) \cup \{\varphi\}) - X = \{\varphi\}$ and since for no $\psi \in (X - (D(\varphi) \cup \{\varphi\}))$ there holds: $\varphi \prec \psi$. Hence $D(\varphi) \cup \{\varphi\} \not\sqsubseteq X$. So X is not an upper bound of $\mathcal{A}$. Contradiction. Therefore $D(\varphi) \cup \{\varphi\}$ is inconsistent. Since $\{\varphi\} = min(D(\varphi) \cup \{\varphi\})$ we have by lemma 5: $P \not\Rightarrow \varphi \in J_\infty$ and $P \subseteq D(\varphi)$. Since $P \subseteq D(\varphi) \subseteq X \subset \Delta_\infty$, $\varphi \notin \Delta_\infty$. Contradiction.

Suppose $|X - \Delta_\infty| \geq 1$. Let $\varphi \in max(X - \Delta_\infty)$. Since $X \sqsubset \Delta_\infty$, there exists a $\psi \in (\Delta_\infty - X)$ with $\varphi \prec \psi$. Hence $\psi \in \Delta_\infty$, $\psi \notin X$ and $\varphi \prec \psi$. Now assume that $D(\psi) \cup \{\psi\}$ is consistent. Then $D(\psi) \cup \{\psi\} \in \mathcal{A}$. Since $(D(\psi) \cup \{\psi\}) - X = \{\psi\}$ and since for no $\eta \in X - (D(\psi) \cup \{\psi\})$ there holds: $\psi \prec \eta$, $D(\psi) \cup \{\psi\} \not\sqsubseteq X$. So X is not an upper bound of $\mathcal{A}$. Contradiction. Hence $D(\psi) \cup \{\psi\}$ must be inconsistent. Since $\{\psi\} = min(D(\psi) \cup \{\psi\})$ we have by lemma 5: $P \not\Rightarrow \psi \in J_\infty$ and $P \subseteq D(\psi)$. Suppose $P \not\subseteq \Delta_\infty$. Then there exists a $\eta \in P$ and $\eta \notin \Delta_\infty$. Since $\eta \in P$, $\varphi \prec \psi \prec \eta$. Because $\eta \in P$, $\eta \in X$ and $\eta \in (X - \Delta_\infty)$. Since $\varphi \in max(X - \Delta_\infty)$, $\varphi \not\prec \eta$. Contradiction. Hence $P \subseteq \Delta_\infty$. Since $P \not\Rightarrow \psi \in J_\infty$, $\psi \notin \Delta_\infty$. Contradiction.

Hence $\Delta_\infty = lub(\mathcal{A})$.

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