A logic for reasoning with inconsistent knowledge

A reformulation using nowadays terminology (2024)

Changes to the original paper The paper is a reformulation of the paper "N.Roos, A logic for reasoning with inconsistent knowledge *Artificial Intelligence* **57** (1992) 69-103" [21] using nowadays terminology. The original paper determines '*justifications*' for deriving conclusions and resolving inconsistencies in provided knowledge and information. These justifications are actually *arguments* that are evaluated using the *stable semantics*, and the approach is an *assumption-based argumentation system*. The current version of the paper uses arguments instead of justifications.

Another change concerns the addition of superscripts to some symbols. The first paragraph of Section 4 states that a linear extension $\prec'$ of the reliability relation $\prec$ is considered. To make this clearer in the formalization, the superscript $^{\prec'}$ is added to anything that depends on the linear extension $\prec'$ that is currently considered.

Section 9 is new and has been inserted to describe the relation with Dung's argumentation framework [5].

The original text of the paper has not been updated except for a few typing errors and improvements to some of the proofs.

History The work on the topic described in the paper is based on the author's Master Thesis (1987) where a ranked set of premisses were used. The ranking was replaced by a partial order in 1988 [16, 17]. In the latter reports, the grounded semantics was used for drawing conclusions and inconsistencies were not resolved in the absence of a unique least preferred premiss among the premisses from which the inconsistency was derived. In 1989, the grounded semantics was replaced by the stable semantics [18, 19, 20] and in the absence of a unique least preferred premiss among the premisses from which the inconsistency is derived, every minimal premiss is considered. [18] was submitted for publication to the *Artificial Intelligence journal* in 1989 after it was rejected for *IJCAI-89*. After a rather long review period, it was accepted with revisions in 1991. The AI journal paper, on which the updated paper presented here is based, was the result of processing the reviewers recommendation.

Related work There is a correspondence between the "logic for reasoning with inconsistent knowledge" and Brewka's "Preferred Subtheories" [2]. Both draw conclusions from preferred maximally consistent subsets of the premisses. The "logic for reasoning with inconsistent knowledge" [16, 17, 18, 19, 20] was developed independent of Brewka's "Preferred Subtheories" [2]. The "logic for reasoning with inconsistent knowledge" differs from Brewka's "Preferred Subtheories" by also presenting an *assumption-based argumentation system* for deriving conclusions, and a *preferential model semantics*.

A logic for reasoning with inconsistent knowledge*
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Abstract

In many situations humans have to reason with inconsistent knowledge. These inconsistencies may occur due to not fully reliable sources of information. In order to reason with inconsistent knowledge, it is not possible to view a set of premisses as absolute truths as is done in predicate logic. Viewing the set of premisses as a set of assumptions, however, it is possible to deduce useful conclusions from an inconsistent set of premisses. In this paper a logic for reasoning with inconsistent knowledge is described. This logic is a generalization of the work of N. Rescher [15]. In the logic a reliability relation is used to choose between incompatible assumptions. These choices are only made when a contradiction is derived. As long as no contradiction is derived, the knowledge is assumed to be consistent. This makes it possible to define an *argumentation*-based deduction process for the logic. For the logic a semantics based on the ideas of Y. Shoham [22, 23], is defined. It turns out that the semantics for the logic is a preferential semantics according to the definition S. Kraus, D. Lehmann and M. Magidor [12]. Therefore the logic is a logic of system **P** and possesses all the properties of an ideal non-monotonic logic.

Keywords: inconsistent information, argumentation, preferential model semantics, assumption-based argumentation system.

*The research reported on was carried out at Delft University of Technology (TU-Delft) [16, 18] and the Royal Netherlands Aerospace Centre (NLR) in Amsterdam [17, 19]. Parts of the research were published in [20]

1 Introduction

In many situations humans have to reason with inconsistent knowledge. These inconsistencies may occur due to sources of information which are not fully reliable. For example, in daylight information about the position of an object coming from your eyes is more reliable than the information about the position of the object coming from your ears. But even reliable sources such as domain experts, do not always agree.

To be able to reason with inconsistent knowledge it is not possible to view a set of premisses as absolute truths, as in predicate logic. Viewing a set of premisses as a set of assumptions, however, makes it possible to deduce useful conclusions from an inconsistent set of premisses. As long as we do not have it proven otherwise, the premisses are assumed to be true statements about the world. When, however, a contradiction is derived, we can no longer make this assumption. To restore consistency, one of the premisses has to be removed. To be able to select a premiss to be removed, a reliability relation on the premisses will be used. This reliability relation denotes the relative reliability of the premisses.

In the following sections I will first describe the propositional case. After describing the propositional case, I will describe how to extend the logic to the first order case.

2 Basic concepts

The language L, that will be used to express the propositions of the logic, consists of the propositions that can be generated using a set of atomic propositions and the logical operators $\neg$ and $\rightarrow$. When in this paper the operators $\wedge$ and $\vee$ are used, they should be interpreted as shortcuts: i.e. $\alpha \wedge \beta$ for $\neg(\alpha \rightarrow \neg\beta)$ and $\alpha \vee \beta$ for $\neg\alpha \rightarrow \beta$.

To be able to reason with inconsistent knowledge, I will consider premisses to be assumptions. These premisses are assumed to be true as long as we do not derive a contradiction from them. If, however, a contradiction is derived, we have to determine the premisses on which the contradiction is based. The premisses on which a contradiction is based are the premisses used in the derivation of the contradiction. When we know these premisses, we have to remove one of them to block the derivation of the contradiction. To select a premiss to be removed, I will use a reliability relation. This reliability relation denotes the relative reliability of the premisses. It denotes that one premiss is more reliable than some other premiss. Clearly the relation must be irreflexive, asymmetric and transitive. I do not demand this relation to be total, for a total reliability relation implies complete knowledge about the relative reliability of the premisses. This does not always have to be the case.

A set of premisses Σ is a subset of the language L. On the set of premisses Σ a partial reliability relation $\prec$ may be defined. Together they form a reliability theory.

Definition 1 A reliability theory is a tuple $\langle \Sigma, \prec \rangle$ where $\Sigma \subseteq L$ is a finite set of premisses and $\prec \subseteq (\Sigma \times \Sigma)$ is an irreflexive, asymmetric and transitive partial reliability relation.

Using the reliability relation, we have to remove a least preferred premiss of the inconsistent set, thereby blocking the derivation of the contradiction.

Example 2 Let Σ denote a set of premisses,

$$\Sigma = \{1.\ \varphi, 2.\ \varphi \to \psi, 3.\ \neg\psi, 4.\ \alpha\}$$

and $\prec$ a reliability relation on Σ:

$$\prec = \{(3,1), (3,2)\}$$

From Σ, ψ can be derived using premisses 1 and 2. Furthermore, a contradiction can be derived from ψ and premiss 3. Hence, the contradiction is based on the premisses 1, 2 and 3. Since premiss 3 is the least reliable premiss on which the contradiction is based, it has to be removed.

Three problems may arise when trying to block the derivation of a contradiction.

- Firstly, we have to be able to determine the premisses on which a contradiction is based. These are the premisses that are used in the derivation of the contradiction. To solve this problem, *supporting arguments* are introduced. A *supporting argument* describes the premisses from which a proposition is derived.

- Secondly, a premiss that has been removed, may have to be placed back because the contradiction causing its removal is also blocked by the removal of another premiss. This may occur because of some other contradiction being derived.

  Example 3 Let Σ be a set of premisses

  $$\Sigma = \{\alpha, \neg\alpha \wedge \neg\beta, \beta\}$$

  and let $\prec$ be a reliability relation on Σ given by

  $$\alpha \prec (\neg\alpha \wedge \neg\beta) \prec \beta.$$

  From α and $\neg\alpha \wedge \neg\beta$ we can derive a contradiction causing the removal of α. From $\neg\alpha \wedge \neg\beta$ and β we can also derive a contradiction causing the removal of $\neg\alpha \wedge \neg\beta$. When $\neg\alpha \wedge \neg\beta$ is removed, it is no longer necessary that α is also removed from the set of premisses to avoid the derivation of a contradiction.

To solve this problem, *undermining arguments*[1] are introduced. An *undermining argument* describes which premiss must be removed if other premisses are assumed to be true. It is a constraint on the set of premisses we assume to be true.

- Thirdly, there need not exist a single least reliable premiss in the set of premisses on which a contradiction is based. This can occur when no reliability relation between premisses is specified. In such a situation we have to consider the results of the removal of every alternative separately.

  Choosing a premiss to be removed implies that we assume the alternatives to be more reliable. Since the reliability relation is transitive, making such a choice influences the reliability relation defined on the premisses.

  Example 4 Let $\Sigma = \{a, b, \neg a, \neg b\}$ be a set of premisses and let $\prec = \{(a, \neg b), (b, \neg a)\}$ be a reliability relation on Σ. Since a and $\neg a$ are in conflict and since there is no reliability relation defined between them, we have to choose a culprit. If we choose to remove $\neg a$, a is assumed to be more reliable. Therefore, $\neg b$ is more reliable than b. Hence, since b and $\neg b$ are also in conflict, b must be removed.

  As is illustrated in the example above, the premisses removed depend on the extension of the reliability relation. Therefore, in the logic described here, every (strict) linear extension of the reliability relation will be considered.

  Different linear extensions of the reliability relation can result in different subsets of the premisses that are assumed to be true statement about the world (*that can be believed*). The set of theorems is defined as the intersection of all extensions of the logic.

As mentioned above, two types of arguments, *supporting arguments* and *undermining arguments*, will be used. A supporting argument is used to denote that a proposition is believed if the premisses in the antecedent are believed, while an undermining argument is used to denote that a premiss can no longer be believed (must be withdrawn) if the premisses in the antecedent are believed.

Definition 5 Let Σ be a set of premisses. Then a *supporting argument* is a formula:

$$P \Rightarrow \varphi$$

where P is a subset of the set of premisses Σ and $\varphi \in L$ is a proposition. $\Rightarrow$ can be viewed as the warrant of the argument [24].
An *undermining argument* is a formula:

$$P \not\Rightarrow \varphi$$

[1] Sometimes the term *undercutting argument* is used. An undercutting argument attacks the application of a defeasible rule. Defeasible premisses can be described by defeasible rules with an empty antecedent.

where P is a subset of the set of premisses Σ and φ is premiss in Σ, but not in P. $\not\Rightarrow$ can be viewed as the warrant of the argument.

3 Characterizing the set of theorems

In this section a characterization, based on the ideas of N. Rescher [15], is given for the set of theorems of a reliability theory. As is mentioned in the previous section, linear extensions of the reliability relation have to be considered. For each linear extension a set of premisses that can still be believed can be determined. This set can be determined by enumerating the premisses with respect to the linear extension of the reliability relation, starting with the most reliable premiss. Starting with an empty set D, if a premiss may consistently be added to the set D, it *should* be added. Otherwise it must be ignored. Because the most reliable premisses are added first, we get a *most reliable consistent set of premisses.*[2]

Definition 6 Let $\langle \Sigma, \prec \rangle$ be a reliability theory. Furthermore, let $\sigma_1, \sigma_2, ..., \sigma_m$ be some enumeration of Σ such that for every $\sigma_j \prec \sigma_k$: $k < j$.

Then D is a most reliable consistent set of premisses if and only if:

$$D = D_m, D_0 = \varnothing$$

and for $0 < i < m$

$$D_{i+1} = \begin{cases} D_i \cup \{\sigma_i\} & \text{if } D_i \cup \{\sigma_i\} \text{ is consistent} \\ D_i & \text{otherwise} \end{cases}$$

Let $\mathcal{R}$ be the set of all the most reliable consistent sets of premisses that can be determined.

Definition 7 Let $\langle \Sigma, \prec \rangle$ be a reliability theory.

Then the set $\mathcal{R}$ of all the most reliable consistent sets of premisses is defined by:

$$\mathcal{R} = \{D \mid \begin{array}{l} D \text{ is a most reliable consistent set of premisses} \\ \text{given some enumeration of } \Sigma \text{ consistent with } \prec \end{array} \}.$$

The set of theorems of a reliability theory is defined as the set of those propositions that are logically entailed by every most reliable consistent set of premisses in $\mathcal{R}$.

[2]The original version of this paper [18, 19] does not consider linear extensions of the reliability relation. The partial ordering of the premisses used in [18, 19] is not a reliability relation, but a preference relation used to choose between incompatible premisses. Considering linear extensions of the reliability relation makes the approach similar to Brewka's *Preferred Subtheories* [2].

Definition 8 Let $\langle \Sigma, \prec \rangle$ be a reliability theory and let $\mathcal{R}$ be the corresponding set of all the most reliable consistent sets of premisses.

Then the set of theorems of $\langle \Sigma, \prec \rangle$ is defined as:

$$Th(\langle \Sigma, \prec \rangle) = \bigcap_{D \in \mathcal{R}} Th(D).$$

where $Th(D) = \{\varphi \mid D \vdash \varphi\}$

4 The deduction process

In this section a deduction process for a reliability theory is described. *Given a strict linear extension $\prec'$ of the reliability relation $\prec$*, the deduction process determines the set of premisses that can be believed.

Remark 9 Instead of starting a deduction process for every strict linear extension of $\prec$, we can also create different extensions of $\prec$ when a contradiction not based on a single least reliable premiss, is derived. This approach results in one deduction tree instead of a deduction sequence for every linear extension of $\prec$.

Instead of deriving new propositions, only new *arguments* are derived. These arguments are generated by the inference rules. The reason why arguments instead of propositions are derived, is that the propositions that can be believed (the belief set) depend on the set of premisses that can still be believed. Since this set of premisses may change because of new information derived, the belief set can change in a non-monotonic way. The arguments, however, do not depend on the information derived. Furthermore, they contain all the information needed to determine the premisses that can still be believed and the corresponding belief set. Note that the set of arguments depends on the linear extension $\prec'$ of $\prec$ that we consider.

Starting with an initial set of arguments $\mathcal{A}_0^{\prec'}$, the deduction process generates a sequence of sets of arguments:

$$\mathcal{A}_0^{\prec'}, \mathcal{A}_1^{\prec'}, \mathcal{A}_2^{\prec'}, ...$$

With each set of arguments $\mathcal{A}_i^{\prec'}$ there corresponds a belief set $B_i^{\prec'}$. So we get a sequence of belief sets:

$$B_0^{\prec'}, B_1^{\prec'}, B_2^{\prec'}, ...$$

Although for the set of arguments there holds:

$$\mathcal{A}_i^{\prec'} \subseteq \mathcal{A}_{i+1}^{\prec'}$$

such a property does not hold for the belief sets. Because a belief set $B_i^{\prec'}$ is determined by evaluating the arguments $\mathcal{A}_i^{\prec'}$, the belief set can change in a non-monotonic way. J. W. Goodwin has called this the process non-monotonicity of the deduction process [10]. According to Goodwin this process non-monotonicity is just another aspect of non-monotonic logics.

In the limit, when all the argument $\mathcal{A}_\infty^{\prec'}$ have been derived, the corresponding belief set B_∞ will be equal to an extension of the reliability theory. Goodwin has called such this process of deriving the set of theorems, the *logical process theory* of a logic [10]. The logical process theory focuses on the deduction process of a logic. In this it differs from the logic itself, which only focuses on derivability; i.e. logics only characterize the set of theorems that follow from the premisses.

A deduction process for the logic starts with an initial set of arguments $\mathcal{A}_0^{\prec'}$. This initial set $\mathcal{A}_0^{\prec'}$ contains a supporting argument for every premiss. These arguments indicate that a proposition is believed if the corresponding premiss is believed.

Definition 10 Let Σ be a set of premisses. Then the set of initial arguments $\mathcal{A}_0^{\prec'}$ is defined as follows:

$$\mathcal{A}_0^{\prec'} = \{\{\varphi\} \Rightarrow \varphi \mid \varphi \in \Sigma\}.$$

Each set of arguments $\mathcal{A}_i^{\prec'}$ with $i > 0$ is generated from the set $\mathcal{A}_{i-1}^{\prec'}$ by adding a new argument. How these arguments are determined, depends on the deduction system used. In the following description of the deduction process, I will use an axiomatic deduction system for the language L, only containing the logical operators $\rightarrow$ and $\neg$.

Axioms The logical axioms are the tautologies of a propositional logic.

Because an axiomatic approach is used, arguments for the axioms have to be introduced. Since an axiom is always valid, it must have an supporting argument with an antecedent equal to the empty set. An axiom is introduced by the following axiom rule.

Rule 1 An axiom φ gets a supporting argument $\varnothing \Rightarrow \varphi$.

In the deduction system two inference rules will be used, namely the modus ponens and the contradiction rule. Modus ponens introduces a new supporting argument for some proposition. This argument is constructed from the arguments for the antecedents of modus ponens.

Rule 2 Let φ and $\varphi \rightarrow \psi$ be two propositions with arguments, respectively $P \Rightarrow \varphi$ and $Q \Rightarrow (\varphi \rightarrow \psi)$.

Then the proposition ψ gets a supporting argument $(P \cup Q) \Rightarrow \psi$.

While modus ponens introduces a new supporting argument, the contradiction rule introduces a new undermining argument to eliminate a contradiction.

Rule 3 Let φ and $\neg\varphi$ be propositions with arguments $P \Rightarrow \varphi$ and $Q \Rightarrow \neg\varphi$ and let $\eta = \min_{\prec'}(P \cup Q)$ where the function *min* selects the minimal element given the extended reliability relation $\prec'$.

Then the premiss η gets an undermining argument $((P \cup Q)/\eta) \not\Rightarrow \eta$.[3]

In order to guarantee that the current set of believed premisses will approximate a most reliable consistent set of premisses, we have to guarantee that the process creating new arguments is fair; i.e. the process does not forever defer the addition of some possible argument to the set of arguments.

Assumption 11 The reasoning process will not defer the addition of any possible argument to the set of arguments forever.

If a fair process is used, the following theorems hold. The first theorem guarantees the soundness of the supporting arguments; i.e. the antecedent of a supporting argument logically entails the consequent of the supporting argument. The second theorem guarantees the completeness of the supporting arguments; i.e. if a proposition is logically entailed by a subset of the premisses, then there exists a corresponding supporting argument. Finally, the third and fourth theorem guarantee respectively the soundness and the completeness of the undermining arguments.

Theorem 12 *Soundness*
For each $i \geq 0$:

if $P \Rightarrow \varphi \in \mathcal{A}_i^{\prec'}$, then:

$P \subseteq \Sigma$ and $P \models \varphi$.

Theorem 13 *Completeness*
For each $P \subseteq \Sigma$:

if $P \models \varphi$, then there exists a $Q \subseteq P$ such that for some $i \geq 0$:

$Q \Rightarrow \varphi \in \mathcal{A}_i^{\prec'}$.

Theorem 14 *Soundness*
For each $i \geq 0$:

if $P \not\Rightarrow \varphi \in \mathcal{A}_i^{\prec'}$, then:

$(P \cup \{\varphi\}) \subseteq \Sigma$, and $(P \cup \{\varphi\})$ is not satisfiable.

[3] In [19, 20], no linear extensions of $\prec$ where considered and an undermining argument for every premiss in $\min_{\prec}(P \cup Q)$ is constructed.

Theorem 15 *Completeness*
For each $P \subseteq \Sigma$:

if P is a minimal unsatisfiable set of premisses and $\varphi = \min_{\prec'}(P)$, where the function *min* selects the minimal element given the extended reliability relation $\prec'$, then for some $i \geq 0$:

$$P \backslash \varphi \not\Rightarrow \varphi \in \mathcal{A}_i^{\prec'}.$$

Given a set of arguments, there exists a set of the premisses that can still be believed. Such a set contains the premisses that do not have to be withdrawn because of an undermining argument. Suppose that $\mathcal{A}_i^{\prec'}$ is a set of arguments derived by a reasoning agent and that $\Delta_i^{\prec'} \subseteq \Sigma$ is the set of the premisses that are assumed to be true by the reasoning agent. Then for each premiss φ such that for some undermining argument $P \not\Rightarrow \varphi \in \mathcal{A}_i^{\prec'}$ there holds that $P \subseteq \Delta_i^{\prec'}$, one may not believe φ. The set of premisses that may not be believed given a set of argument $\mathcal{A}_i^{\prec'}$, is denoted by $Out_i^{\prec'}(\Delta_i^{\prec'})$.

Definition 16

$$Out_i^{\prec'}(S) = \{\varphi \mid P \not\Rightarrow \varphi \in \mathcal{A}_i^{\prec'}, \text{ and } P \subseteq S\}$$

The set of premisses $\Delta_i^{\prec'}$ must, of course, be equal to the set of premisses obtained after removing all the premisses we may not believe; i.e. $\Delta_i^{\prec'} = \Sigma \backslash Out_i^{\prec'}(\Delta_i^{\prec'})$. The set of premisses that satisfy this requirement is defined by the following fixed point definition.

Definition 17 Let Σ be a set of premisses and let $\mathcal{A}_i^{\prec'}$ be a set of arguments. Then the set of premisses $\Delta_i^{\prec'}$ that can be assumed to be true, is defined as:

$$\Delta_i^{\prec'} = \Sigma \backslash Out_i^{\prec'}(\Delta_i^{\prec'}).$$

Property 18 For every i, the set $\Delta_i^{\prec'}$ exists and is unique.

After determining the set of premisses that can be believed, the set of derived propositions that can be believed can be derived from the supporting arguments. This set is defined as:

Definition 19 Let $\mathcal{A}_i^{\prec'}$ be a set of arguments and $\Delta_i^{\prec'}$ be the corresponding set of premisses that may assumed to be true.
The set of propositions $B_i^{\prec'}$ that can be believed (*the belief set*) is defined as:

$$B_i^{\prec'} = \{\psi \mid \ P \Rightarrow \psi \in \mathcal{A}_i^{\prec'} \text{ and } P \subseteq \Delta_i^{\prec'}\}.$$

Property 20 For each $\varphi \in B_i^{\prec'}$: $\Delta_i^{\prec'} \vdash \varphi$.

Let $\mathcal{A}_\infty^{\prec'}$ be the set of all arguments that can be derived.

Definition 21 $\mathcal{A}_\infty^{\prec'} = \bigcup_{i \geq 0} \mathcal{A}_i^{\prec'}$

The corresponding set of premisses that can be believed and the belief set, will be denoted by respectively $\Delta_\infty^{\prec'}$ and by $B_\infty^{\prec'}$.

Property 22 $\Delta_\infty^{\prec'}$ is maximal consistent.

Property 23

$$B_\infty^{\prec'} = Th(\Delta_\infty^{\prec'})$$

where $Th(S) = \{\varphi \mid S \vdash \varphi\}$

The following theorem implies that the characterization of the theorems of the logic, given in the previous section, is equivalent to the intersection of the belief sets that can be derived.

Theorem 24 Let $\langle \Sigma, \prec \rangle$ be a reliability theory.

Then there holds:

$$\mathcal{R} = \{\Delta_\infty^{\prec'} \mid \text{ for some linear extension } \prec' \text{ of } \prec,\ \Delta_\infty^{\prec'} \text{ can be derived}\}.$$

Corollary 25

$$Th(\langle \Sigma, \prec \rangle) = \bigcap \{B_\infty^{\prec'} \mid \text{ for some linear extension } \prec' \text{ of } \prec\}.$$

5 Determination of the belief set

In this section I will describe the algorithms that determine the set of premisses that can be believed and the belief set, given a set of undermining arguments. The first algorithm determines the set $\Delta_i^{\prec'}$ given the arguments $\mathcal{A}_i^{\prec'}$. To understand how the algorithm works, recall that the consequent of an undermining argument is less reliable than the premisses in the antecedent. Therefore, if the consequent of an undermining argument $P \not\Rightarrow \varphi$ is the most reliable premiss that can be remove, because we still belief the premisses in the antecedent P, removing φ will never have to be undone. After having removed φ we can turn to the next most reliable consequent of an undermining argument.

The time complexity of the algorithm below depends on the **for** and the **repeat** loop. The former loop can be executed in $\mathcal{O}(n)$ steps where n in the number of undermining arguments. The latter loop can be executed in $\mathcal{O}(m)$ steps where m in the number of premisses in Σ. Therefore, the whole algorithm can be executed in $\mathcal{O}(n \cdot m)$ steps.

begin
  $\Delta_i^{\prec'} := \Sigma$;
  repeat
    $\varphi \in max(\Sigma)$;
    $\Sigma := \Sigma \backslash \varphi$;
    for each $P \not\Rightarrow \varphi \in \mathcal{A}_i^{\prec'}$ **do**
      if $P \subseteq \Delta_i^{\prec'}$
      then $\Delta_i^{\prec'} := \Delta_i^{\prec'} \backslash \varphi$;
  until $\Sigma = \varnothing$;
  return $\Delta_i^{\prec'}$;
end.

Using the supporting arguments, the belief set $B_i^{\prec'}$ can be determined in a straightforward way. Clearly, $B_i^{\prec'}$ can be determined in $\mathcal{O}(n)$ steps where n is the number of supporting arguments.

begin
  $B_i^{\prec'} = \varnothing$;
  repeat
    $P \Rightarrow \varphi \in \mathcal{A}_i^{\prec'}$;
    $\mathcal{A}_i^{\prec'} := \mathcal{A}_i^{\prec'} \backslash \{P \Rightarrow \varphi\}$;
    if $P \subseteq \Delta_i^{\prec'}$
    then $B_i^{\prec'} := B_i^{\prec'} \cup \{\varphi\}$;
  until $\mathcal{A}_i^{\prec'} = \varnothing$;
  return $B_i^{\prec'}$;
end.

6 The semantics for the logic

The semantics of the logic is based on the ideas of Y. Shoham [22, 23]. In [22, 23] Shoham argues that the difference between monotonic logic and non-monotonic logic is a difference in the definition of the entailment relation. In a monotonic logic a proposition is entailed by the premisses if it is true in every model for the premisses. In a non-monotonic logic, however, a proposition is entailed by the premisses if it is preferentially entailed by a set of premisses; i.e. if it is true in every preferred model for the premisses. These preferred models are determined by defining an acyclic partial preference order on the models.

The semantics for the logic differs slightly from Shoham's approach. Since the set of premisses may be inconsistent, the set of models for these premisses can be empty. Therefore, instead of defining a preference relation on the models of the premisses, a partial preference relation on the set of semantic interpretations for the language is defined. Given such a preference relation on the interpretations, the models for a reliability theory are the most preferred semantic interpretations. The preference relation used here is based on the following ideas.

- The premisses are assumptions about the world we are reasoning about.
- We are more willing to give up believing a premiss with a low reliability than a premiss with a high reliability.

Therefore, an interpretation satisfying more premisses with a higher reliability ($\prec$) than some other interpretation, is preferred ($\sqsubset$) to this interpretation.

Example 26 Let $\mathcal{M}$ and $\mathcal{N}$ be two interpretations. Furthermore , let $\mathcal{M}$ satisfy α and β, and let $\mathcal{N}$ satisfy β and γ. Finally let α be more reliable than γ, $\gamma \prec \alpha$. Clearly, we cannot compare $\mathcal{M}$ and $\mathcal{N}$ using the premiss β. $\mathcal{M}$ and $\mathcal{N}$ can, however, be compared using the premisses α and γ. Since α is more reliable than γ, since $\mathcal{N}$ does not satisfy α and since $\mathcal{M}$ does not satisfy γ, $\mathcal{M}$ must be preferred to $\mathcal{N}$,

Definition 27 An interpretation $\mathcal{M}$ is a set containing the atomic propositions that are true in this interpretation.

Definition 28 Let $\mathcal{M}$ be a semantic interpretation and let Σ be a set of premisses. Then the premisses $Prem(\mathcal{M}) \subseteq \Sigma$ that are satisfied by $\mathcal{M}$, are defined as:

$$Prem(\mathcal{M}) = \{\varphi \mid \varphi \in \Sigma \text{ and } \mathcal{M} \models \varphi\}$$

Definition 29 Let $\langle \Sigma, \prec \rangle$ be a reliability theory. Furthermore, let $\sqsubset$ be a preference relation on the interpretations.
For every interpretation $\mathcal{M}, \mathcal{N}$ there holds:

$\mathcal{M} \sqsubset \mathcal{N}$ if and only if $Prem(\mathcal{M}) \neq Prem(\mathcal{N})$ and for every $\varphi \in (Prem(\mathcal{M}) \backslash Prem(\mathcal{N}))$, there is a $\psi \in (Prem(\mathcal{N}) \backslash Prem(\mathcal{M}))$ such that:

$$\varphi \prec \psi.$$

The preference relation on the interpretations has the following property:

Property 30 Let $\langle \Sigma, \prec \rangle$ be a reliability theory and let $\sqsubset$ be the preference relation over interpretations defined the reliability theory.

$\sqsubset$ is irreflexive and transitive.

Given the preference relation on the interpretations, the set of models for the premisses can be defined.

Definition 31 Let $\langle \Sigma, \prec \rangle$ be a reliability theory and let $Mod_{\sqsubset}(\langle \Sigma, \prec \rangle)$ denote the models for the reliability theory.

$\mathcal{M} \in Mod_{\sqsubset}(\langle \Sigma, \prec \rangle)$ if and only if there exists no interpretation $\mathcal{N}$ such that:

$$\mathcal{M} \sqsubset \mathcal{N}.$$

Now the following important theorem, guaranteeing the soundness and the completeness of the logic, holds:

Theorem 32 Let $\langle \Sigma, \prec \rangle$ be a reliability theory. Furthermore, let $\mathcal{R}$ be the corresponding set of all most reliable consistent sets of premisses. Then:

$$Mod_{\sqsubset}(\langle \Sigma, \prec \rangle) = \bigcup_{\Delta_\infty^{\prec'} \in \mathcal{R}} Mod(\Delta_\infty^{\prec'})$$

where $Mod(S)$ denotes the set of classical models for a set of propositions S.

7 Some properties of the logic

In this section I will discuss some properties of the logic. Firstly, I will relate the logic to the general framework for non-monotonic logics described by S. Kraus, D. Lehmann and M. Magidor [12]. Secondly, I will compare the behaviour of the logic when new information is added with Gärdenfors's theory for belief revision [8].

7.1 Preferential models and cumulative logics

In [12] Kraus et al. describe a general framework for the study of non-monotonic logics. They distinguish five general logical systems and show how each of them can be characterized by the properties of the consequence relation. Furthermore, for each consequence relation a different class of models is defined. The consequence relations and the classes of models are related to each other by representation theorems.

The consequence relation relevant for the logic discussed here is the preferential consequence relation of system **P**. I will show that the preference relation on the semantic interpretations, described in the previous section, corresponds to a preferential model described by Kraus et al.

Lemma 33 Let $\langle \Sigma, \prec \rangle$ be a reliability theory. Furthermore, let $\widehat{\alpha} = \{\mathcal{M} \mid \mathcal{M} \models \alpha\}$, let $\Sigma' = \Sigma \cup \{\alpha\}$ and let $\prec' = (\prec \cap (\Sigma \backslash \alpha \times \Sigma \backslash \alpha)) \cup \{\langle \varphi, \alpha \rangle \mid \varphi \in \Sigma \backslash \alpha\}$.

Then $\mathcal{M} \in Mod_{\sqsubset'}(\langle \Sigma', \prec' \rangle)$ if and only if $\mathcal{M} \in \widehat{\alpha}$ and for no $\mathcal{N} \in \widehat{\alpha}$:

$$\mathcal{M} \sqsubset \mathcal{N}.$$

Theorem 34 Let $\langle \Sigma, \prec \rangle$ be a reliability theory. Moreover, let $\langle S, l, < \rangle$ be a triple where the set of states S is the set of all possible interpretations for the language L, where $l : S \to S$ is the identity function, and where for each $\mathcal{M}, \mathcal{N} \in S$:

$\mathcal{M} < \mathcal{N}$ if and only if $\mathcal{N} \sqsubset \mathcal{M}$.

Then $\langle S, l, < \rangle$ is a *preferential model* [12].

Now I will relate the consequence relation of system **P** to the logic. To motivate the relation I will describe below, recall that $\alpha \mathrel{|\!\sim} \beta$ should be interpreted as: 'if α, normally β'. Hence, if we assume α, we must assume that α is true beyond any doubt. To realize this, we must add α as a premiss. Furthermore, α must be more reliable than any other premiss, otherwise we cannot guarantee that α is an element of the set of theorems $Th(\langle \Sigma, \prec \rangle)$. It is possible that α is an element of the original set of premisses. In that case we should use the most reliable knowledge source for a premiss; i.e. the assumption that α is true beyond any doubt. If α is indeed an element of B_∞, we must prove that β will also be an element of $Th(\langle \Sigma, \prec \rangle)$.

Theorem 35 Let $W = \langle S, l, < \rangle$ be a preferential model for $\langle \Sigma, \prec \rangle$. Then the following equivalence holds:

$\alpha \mathrel{|\!\sim}_W \beta$ if and only if

$$\Sigma' = \Sigma \cup \{\alpha\},$$

$$\prec' = (\prec \cap (\Sigma \backslash \alpha \times \Sigma \backslash \alpha)) \cup \{\langle \varphi, \alpha \rangle \mid \varphi \in \Sigma \backslash \alpha\}$$

and $\beta \in Th(\langle \Sigma', \prec' \rangle)$.

Corollary 36 Let $W = \langle S, l, < \rangle$ be a preferential model for $\langle \Sigma, \prec \rangle$. Then:

$$Th(\langle \Sigma, \prec \rangle) = \{\alpha \mid \ \mathrel{|\!\sim}_W \alpha\}$$

7.2 Belief revision

In [8], Gärdenfors describes three different ways in which a belief set can be revised, viz. *expansion, revision* and *contraction.* Expansion is a simple change that follows from the addition of a new proposition. Revision is a more complex form of adding a new proposition. Here the belief set must be changed in such a way that the resulting belief set is consistent. Contraction is the change necessary to stop believing some proposition. For each of these forms of belief revision, Gärdenfors has formulated a set of *rationality postulates.*

In this subsection I will investigate which of the postulates are satisfied by the logic. To be able to do this, the set of theorems of a reliability theory is identified as a belief set as defined by Gärdenfors. Here expansion, revision and contraction of the belief set K, with respect to the proposition α, will be denoted by respectively: $K^+[\alpha]$, $K^*[\alpha]$ and $K^-[\alpha]$.

Expansion

To expand a belief set K with respect to a proposition α, α should be added to the set of premisses that generate the belief set. Since the logic does not allow an inconsistent belief set, α can be added if the belief set does not already contain $\neg\alpha$. Otherwise, the logic would start revising the belief set. Adding α to the set of premisses, however, is not sufficient to guarantee that α will belong to the new belief set. Take for example the following reliability theory.

$$\Sigma = \{1 : \alpha \wedge \beta, 2 : \neg\alpha \wedge \beta, 3 : \alpha \wedge \neg\beta, 4 : \neg\alpha \wedge \neg\beta\}$$

$$\prec = \{(3, 2), (4, 1)\}$$

Clearly, adding α to Σ does not result in believing α. Hence, the second postulate of expansion is not satisfied. To guarantee that α belongs to the new belief set, we have to prefer α to any other premiss. If, however, we prefer α to every other premiss in the example above, the third postulate for expansion will not be satisfied. Hence, expansion of a belief set is not possible in the logic. The reason for this is that the reasons for believing a proposition in a belief set are not taken into account by the postulates for expansion. Because of this internal structure, revision instead of expansion takes place.

Revision

For revision of a belief set K with respect to a proposition α, we have to add α as a premiss and prefer it to any other premiss. With this implementation of the revision process, some of the postulates for revision of the belief set with respect to α are satisfied. The postulates not being satisfied relate revision to expansion. Expansion, however, is not defined for the logic.

Theorem 37 Let belief set $K = Th(\langle\Sigma, \prec\rangle)$ be the set of theorems of the reliability theory $\langle\Sigma, \prec\rangle$.
Suppose that $K^*[\alpha]$ is the belief set of the premisses $\Sigma \cup \{\alpha\}$ with reliability relation:

$$\prec' = (\prec \cap (\Sigma\backslash\alpha \times \Sigma\backslash\alpha)) \cup \{\langle\varphi, \alpha\rangle \mid \varphi \in \Sigma\backslash\alpha\};$$

i.e. $K^*[\alpha] = \{\beta \mid \alpha \mathrel{|\!\sim}_W \beta\}$ where W is a preferential model for $\langle\Sigma, \prec\rangle$.
Then the following postulates are satisfied.

1. $K^*[\alpha]$ is a belief set.
2. $\alpha \in K^*[\alpha]$.
6. If $\vdash \alpha \leftrightarrow \beta$, then $K^*[\alpha] = K^*[\beta]$.

Contraction

It is not possible to realise contraction of a belief set in the logic in a straight forward way. To be able to contract a proposition α from a belief set K, we have to determine the premisses on which belief in this proposition is based. This information can be found in the applicable supporting argument that supports the proposition α. When we have determined these premisses, we have to remove some of them. I.e. for each linear extension of the reliability relation, we must add the following undermining arguments to $\mathcal{A}_\infty^{\prec'}$

$$\{P\backslash\varphi \not\Rightarrow \varphi \mid P \Rightarrow \alpha \in \mathcal{A}_\infty^{\prec'}, \varphi = \min_{\prec'}(P)\}.$$

Unfortunately, this solution, which requires a modification of the logic, can only be applied after $\mathcal{A}_\infty^{\prec'}$ has been determined. Furthermore, only the most trivial postulates 1, 3, 4 and 6 will be satisfied.

8 Extension to first order logic

The logic described in the previous sections can be extended to a first order logic. To realize this we have to replace the propositional language L by a first order language, which only contains the logical operators $\neg$ and $\rightarrow$, and the quantifier $\forall$. Furthermore we have to replace the logical axioms for a propositional logic by the logical axioms for a first order logic with the modus ponens as the only inference rule. We can for example use the following axiom scheme, which originate from [7].

Axioms Let φ be a generalization of ψ if and only if for some $n \geq 0$ and variables $x_1, ..., x_n$:

$$\varphi = \forall x_1, ..., \forall x_n\ \psi.$$

Since this definition includes the case $n = 0$, any formula is a generalization of itself.

The logical axioms are all the generalizations of the formulas described by the following schemata.

1. Tautologies.
2. $\forall x\varphi(x) \rightarrow \varphi(t)$ where t is a term containing no variables that occur in φ.
3. $\forall x(\varphi \rightarrow \psi) \rightarrow (\forall x\varphi \rightarrow \forall x\psi)$.
4. $\varphi \rightarrow \forall x\varphi$ where x does not occur in φ.

Finally, we have to replace the definition of the semantic interpretations by a definition for the semantic interpretations of a first order logic.

When these modification are made we have a first order logic for reasoning with inconsistent knowledge. For this first order logic all the results that can be found in the preceding section also hold.

9 Argumentation framework (a new section)

Dung [5] observed that argumentation systems proposed in the literature, use the same types of semantics and that these semantics can be studied independent of the underlying argumentation system. He also showed that several other forms of non-monotonic reasoning can be reformulated as argumentation systems. To study the different semantics of argumentation systems independent of the underlying argumentation system, he introduces the notion of an *argumentation framework*.

An argumentation framework is a tuple

$$\langle \mathcal{A}, \longrightarrow \rangle$$

where $\mathcal{A}$ is a set of arguments and $\longrightarrow \subseteq \mathcal{A} \times \mathcal{A}$ is a attack relation over the arguments. The relation $\longrightarrow$ denotes for every $(A, B) \in \longrightarrow$ that the argument B cannot be valid if A is valid. How the arguments $\mathcal{A}$ and the attack relation over the arguments are derived and what is supported by the arguments is not taken into consideration. Note that instead of $(A, B) \in \longrightarrow$, below I will use the infix notation $A \longrightarrow B$.

Section 2 defined two types of arguments. Only the undermining arguments of the form $P \not\Rightarrow \varphi$ attack other arguments. Given an set of arguments $\mathcal{A}_i^{\prec'}$, an argumentation framework $\langle \mathcal{A}_i^{\prec'}, \longrightarrow_i^{\prec'} \rangle$ can be defined.

Definition 38 Let $\mathcal{A}$ be a set of arguments.

$\langle \mathcal{A}, \longrightarrow \rangle$ is a corresponding argumentation framework where $A \longrightarrow B$ iff

- $\{A, B\} \subseteq \mathcal{A}$,
- $A = (P \not\Rightarrow \varphi)$,
- $B = (Q \not\Rightarrow \psi)$ or $B = (Q \Rightarrow \psi)$, and
- $\varphi \in Q$.

Note that the argumentation framework is an instance of *assumption-based argumentation* [6].

Because we are considering a linear extension $\prec'$ of $\prec$, which is a total ordering of Σ, there is a unique stable argument extension $\mathcal{E}_i^{\prec'}$ for an argumentation framework $\langle \mathcal{A}_i^{\prec'}, \longrightarrow_i^{\prec'} \rangle$, which is also the unique grounded extension. This stable extension $\mathcal{E}_i^{\prec'}$ determines the premisses $\Delta_i^{\prec'}$ that can be assumed to be true:

$$\Delta_i^{\prec'} = \Sigma \backslash \{\varphi \mid P \not\Rightarrow \varphi \in \mathcal{E}_i^{\prec'}\}$$

as well as the belief set:

$$B_i^{\prec'} = \{\varphi \mid P \Rightarrow \varphi \in \mathcal{E}_i^{\prec'}\}$$

In the original version [18, 19] of the AI journal paper [21], no linear extensions $\prec'$ of $\prec$ was considered. Instead, whenever two supporting arguments support a proposition and its negation, for every least preferred supporting premiss given the partial order $\prec$ on the premisses, an undermining argument is formulated.

Rule 4 Let φ and $\neg\varphi$ be propositions with arguments $P \Rightarrow \varphi$ and $Q \Rightarrow \neg\varphi$.

Let $\eta \in \min_{\prec}(P \cup Q)$ be a minimal element given the reliability relation $\prec$.

Then the premiss η gets an undermining argument $((P \cup Q)/\eta) \not\Rightarrow \eta$.

So, if there is no unique least preferred premiss in $P \cup Q$ given $\prec$, multiple undermining arguments are formulated. Moreover, the stable semantics may result in multiple argument extensions. Some of these argument extensions may determine a set of premisses Δ_i, but give rise to the problem illustrated in Example 4. That is, selecting a minimal element in $\min_{\prec}(P \cup Q)$ introduces a preference, and all the introduced preferences combined with $\prec$ do not correspond to any linear extension of $\prec$ because the combination contains cycles.

Instead of considering all linear extensions of $\prec$ as was described in Section 4, we can also apply Rule 4 and determine all stable argument extensions. Some of these stable extensions may not correspond to a linear extension of $\prec$ and have to be ignored. An argument extension has to be ignored if $\prec$ together with the additional preferences introduced by the argument extension contains a cycle. Formally:

Definition 39 Let $\prec$ be a partial order on the premisses Σ, and let $\mathcal{E}$ be an argument extension.

The argument extension $\mathcal{E}$ must be *ignored* if and only if the partial order

$$\prec^* = \prec \cup \{(\varphi, \psi) \mid P \not\Rightarrow \varphi \in \mathcal{A}, \psi \in P, P \subseteq \mathcal{E}\}$$

over Σ contains a cycle.

Note that there exists at least one linear extension $\prec^*$ of $\prec$ if $\prec$ contains no cycles. This approach is more efficient than considering all linear extensions of $\prec$, which has a worst case time complexity that is factorial in the number of premisses Σ.

10 Related work

In this section I will discuss some related approaches. Firstly, the relation with of N. Rescher's work will be discussed. Rescher's work is closely related to the logic described here. Secondly, the relation with Poole's framework for default reasoning, which is a special case of Rescher's work, will be discussed. Thirdly, the difference between paraconsistent logics and the logic described here, will be discussed. Finally the relation with Truth Maintenance Systems, and especially J. de Kleer's ATMS will be discussed.

10.1 Hypothetical reasoning

In his book ‘Hypothetical Reasoning’, Rescher describes how to reason with an inconsistent set of premisses [15]. He introduces his reasoning method, because he wants to formalize hypothetical reasoning. In particular, he wants to formalize reasoning with belief contravening hypotheses, such as counterfactuals. In the case of counterfactual reasoning, we make an assumption that we know to be false. For example, let us suppose that Plato had lived during the middle ages. To be able to make such a counter factual assumption, we, temporally, have to give up some of our beliefs to maintain consistency. It is, however, not always clear which of our beliefs we have to give up. The following example gives an illustration.

Example 40

Beliefs

1. Bizet was of French nationality.
2. Verdi was of Italian nationality.
3. Compatriots are persons who share the same nationality.

Hypothesis Assume that Bizet and Verdi are compatriots.

There are three possibilities to restore consistency. Clearly, we do not wish to witdraw 3, but we are indifferent whether we should give up 1 or 2.

To model this behaviour in a logical system, Rescher divides the set of premisses into modal categories. The modalities Rescher proposes are: alethic modalities, epistemic modalities, modalities based on inductive warrant, and modalities based on probability or confirmation. Given a set of modal categories, he selects Preferred Maximal Mutually-Compatible subsets (PMMC subsets) from them. The procedure for selecting these subsets is as follows:

Let $M_0, ..., M_n$ be a family of modal categories.

1. Select a maximal consistent subset of M_0 and let this be the set S_0.
2. Form S_i by adding as many premisses of M_i to S_{i-1} as possible without disturbing the consistency of S_i.

S_n is a PMMC-subset.

Given these PMMC-subsets, Rescher defines two entailment relations.

- Compatible-Subset (CS) entailment. A proposition is CS entailed if it follows from every PMMC-subset.
- Compatible-Restricted (CR) entailment. A proposition is CR entailed if it follows from some PMMC-subset.

It is not difficult to see that Rescher's modal categories can be represented by a partial reliability relation on the premisses. For every modal category M_i, M_j with $i < j$, there must hold that each premiss in M_i is more reliable than any premiss in M_j. Given this ordering, from Definition 6 it follows immediately that the PMMC-subsets are equal to the most reliable consistent sets of premisses.

10.2 A framework for default reasoning

The central idea behind Poole's approach is that default reasoning should be viewed as *scientific theory formation* [13]. Given a set of facts about the world and a set of hypotheses, a subset of the hypotheses which together with the facts can explain an *observation*, have to be selected. Of course, this selected set of hypotheses has to be consistent with the facts. A default rule is represented in Poole's framework by a hypothesis containing free variables. Such a hypothesis represents a set of ground instances of the hypothesis. Each of these ground instances can be used independently of the other instances in an explanation. An explanation for a proposition φ is a maximal (with respect to the inclusion relation) *scenario* that implies φ. Here a scenario is a consistent set containing all the facts and some ground instances of the hypotheses.

This framework can be viewed as a special case of Rescher's work. Poole's framework consists of only two modal categories, the facts M_0 and the hypotheses M_1. Since Rescher's work is a special case of the logic described in this paper, so is Poole's framework. Poole, however, extends his framework with constraints. These constraints are added to be able to eliminate some scenarios as possible explanations for a formula φ. A scenario is eliminated when it is not consistent with the constraints.

The constraints can be interpreted as describing that some scenarios are preferred to others. Since in the logic described in this paper a reliability relation on the premisses generates a preference relation on consistent subsets of the premisses, an obvious question is whether the preference relation described by the constraints can be modelled with an appropriate reliability relation. Unfortunately, the answer is 'no'. This is illustrated by the following example.

Example 41

Facts: φ, ψ.

Defaults: $\varphi \to \alpha, \varphi \to \neg\beta, \psi \to \neg\alpha, \psi \to \beta$.

Constraints: $\neg(\alpha \wedge \beta), \neg(\neg\alpha \wedge \neg\beta)$.

Without the constraints this theory has four different extensions. These extensions are the logical consequences of the following scenarios.

$$S_1 = \{\varphi, \psi, \varphi \to \alpha, \varphi \to \neg\beta\}$$

$S_2 = \{\varphi, \psi, \psi \to \neg\alpha, \psi \to \beta\}$

$S_3 = \{\varphi, \psi, \varphi \to \alpha, \psi \to \beta\}$

$S_4 = \{\varphi, \psi, \varphi \to \neg\beta, \psi \to \neg\alpha\}$

Only the first two scenarios are consistent with constraints. If this default theory has to be modelled in the logic, a reliability relation has to be specified in such a way that $\{S_1, S_2\} = \mathcal{R}$. To determine the required reliability relation on the hypotheses, combinations of two scenarios are considered. To ensure that $S_1 \in \mathcal{R}$ and $S_3 \notin \mathcal{R}$, $\varphi \to \neg\beta$ has to be more reliable than $\psi \to \beta$. To ensure that $S_2 \in \mathcal{R}$ and $S_4 \notin \mathcal{R}$, $\psi \to \beta$ has to be more reliable than $\varphi \to \neg\beta$. Hence, the reliability relation would be reflexive, violating the requirement of irreflexivity in a strict partial order. This means that not every ordering of explanations in Poole's framework can be modelled, using the logic described in this paper.

Although Poole's framework without constraints can be expressed in the logic described in this paper, the philosophies behind the two approaches are quite different. Poole's work is based on the idea that default reasoning is a process of selecting consistent sets of hypotheses, which can explain a set of observations. The purpose of the logic described in this paper, however, is to derive useful conclusions from an inconsistent set of premisses.

10.3 Paraconsistent logics

Paraconsistent logics are a class of logics developed for reasoning with inconsistent knowledge [1]. Unlike classical logics, in paraconsistent logics there need not hold $\neg(\varphi \wedge \neg\varphi)$ for some proposition φ. Hence, an inconsistent set of premisses is not equivalent to the trivial theory; it does not imply the set of all propositions.

Unlike the logic described in this paper, a paraconsistent logic does not resolve an inconsistency. Instead it simply avoids that everything follows from an inconsistent theory. To illustrate this more clearly, consider the following a reliability theory, without a reliability relation.

$$\Sigma = \{\alpha \wedge \beta, \neg\beta \wedge \gamma\}$$

In the logic described in this paper, all maximal consistent subsets will be generated.

$$\{\alpha \wedge \beta\} \text{ and } \{\neg\beta \wedge \gamma\}$$

In a paraconsistent logic the proposition β will be contradictory but the propositions α and γ will consistently be entailed by the premisses.

The difference between the two approaches can be interpreted as the difference between a credulous and a sceptical view of knowledge sources. With a credulous view of a knowledge source, we try to derive as much as is consistently possible. According to Arruda [1], scientific theories for different domains, which conflict with each other on some overlapping aspect, are treated in this way. With a sceptical view of a knowledge source, we only believe one of the knowledge sources that support the conflicting information. So if a part of someone statement turns out to be wrong, we will not belief the rest of his/her statement. Although a credulous view of knowledge sources seems to be acceptable for scientific theories for different domains, a sceptical view seems to be better for knowledge based systems, which have to act on the information available.

10.4 Truth maintenance systems

In the here presented logic arguments are used. Justifications in the JTMS of J. Doyle [4] or the ATMS of J. de Kleer [11] have a similar function as arguments. Unlike the arguments used here, these justifications are not part of the deduction process. The arguments used here follow directly from the requirement for the deduction process (Section 2). Moreover, in a(n) (A)TMS the justifications describe dependencies between propositions, while in the here presented logic, the supporting arguments describe dependencies between propositions and premisses, and undermining arguments describe dependencies among premisses. The supporting arguments of the logic, however, can be compared with the labels in the ATMS [11]. Like a label, a supporting argument describes from which premisses a proposition is derived. The undermining arguments have more or less the same function as a **nogood** in the ATMS. As with an element from the set representing a **nogood**, the consequent and the antecedents of an undermining argument may not be assumed to be true simultaneously. Unlike an element of the set nogood, an undermining argument describes which element has to be removed from the set of premisses (assumptions).

Because supporting arguments and labels are closely related, it is possible to describe an ATMS using a propositional logic. Let $\langle A, N, J\rangle$ be an ATMS where:

- A is a set of assumptions,
- N is a set of nodes, and
- J is a set of justifications.

We can model the ATMS in the logic using the following construction. Let $A \cup N$ be the set of atomic propositions of the logic. Furthermore, let the set of premisses Σ be equal to $A \cup J$ where the justifications J are described by rules of the form:

$$p_1 \wedge ... \wedge p_n \to q.$$

Finally, let every justification be more reliable than any assumption. Then the set $\mathcal{R}$ is equal to the set of maximal (under the inclusion relation) environments of an ATMS. Furthermore, for any linear extension of the reliability relation, the label for a node $n \in N$ is equal to the set:

$$\{P \mid P \Rightarrow n \in \mathcal{A}_\infty^{\prec'} \text{ and for no } Q \Rightarrow n \in \mathcal{A}_\infty^{\prec'}\text{: } Q \subset P\}.$$

The set of nogoods is equal to the set:

$$\{(P \cup \{p\}) \cap A \mid \; P \not\Rightarrow p \in \mathcal{A}_\infty^{\prec'} \text{ and for no } Q \not\Rightarrow q \in \mathcal{A}_\infty^{\prec'}\text{: } (Q \cup \{q\}) \cap A \subset (P \cup \{p\}) \cap A\}.$$

11 Applications

In the previous sections a logic for reasoning with inconsistent knowledge was described. In this section two applications will be discussed.

Unreliable knowledge sources

In situations where we must deal with sensor data the logic described in the previous sections can be applied. To be able to reason with sensor data, the data has to be translated into statements about the world. Because of measurements errors and of misinterpretation of the data, these statements can be incorrect. This may result in inconsistencies. These inconsistencies may be resolved by considering the reliability of the knowledge sources used. To illustrate this consider the following example.

Example 42 Suppose that we want to determine the type of an airplane by using the characteristic of its radar reflection. The radar reflection of an airplane depends on the size and the shape of plane. Suppose that we have some pattern recognition system that outputs a proposition stating the type of plane, or a disjunction of possible types in case of uncertainty. Furthermore, suppose that we have an additional system that determines the speed and the course of the plane. The output of this system will also be stated as a proposition. Given the output of the two systems, we can verify whether they are compatible. If a plane is recognized as a Dakota and its speed is 1.5 Mach, then, knowing that a Dakota cannot go through the sound barrier, we can derive a conflict. Since the speed measuring system is more reliable than the type identifying system, we must remove the proposition stating that the plane is a Dakota.

In this example, the reliability relation can be interpreted as denoting that if two premisses are involved in a conflict the least reliable premiss has the highest probability of being wrong. Since the relative probability is conditional on inconsistencies, information from one reliable source cannot be overruled by information from many unreliable knowledge sources. For example, the position of an object determined by seeing it is normally more reliable then the position determined hearing it, independent of the number persons that heard it at some position. Notice that fault probabilities have no meaning because faults are context dependent. The positions where you hear an object can be incorrect because of reflections and the limited speed of sound. Usually, these factors cannot be predicted in advance.

Planning

Another possible application for the logic can be found in the area of planning. In [9], Ginsberg and Smith describe a possible worlds approach for reasoning about actions. What they propose is an alternative way of determining the consequences of an action. Instead of using frame axioms, default rules, or add and delete lists. They propose to determine the nearest *world* that is consistent with the consequences of an action. The advantage of this approach is that we do not have to know all possible consequences of an action in advance. For example, in general, we cannot know whether putting a plant on a table will obscure a picture on the wall. Hence, if we know that a picture is not obscured before an action, we may assume that it is still not obscured after the action when this fact is consistent with the consequences of the action.

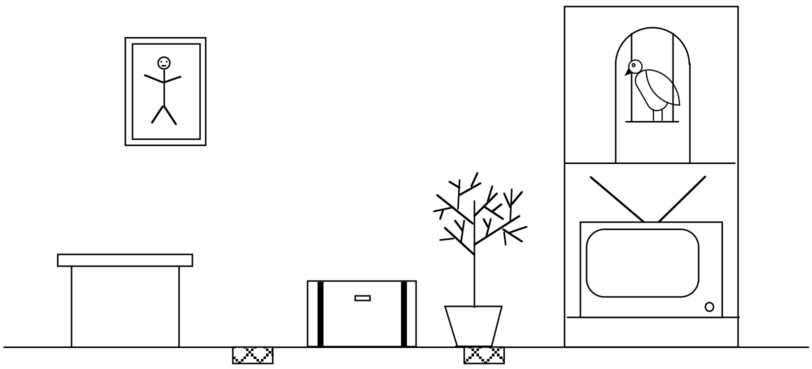

Figure 1: living-room

Example 43 Figure 1 can be described a set of premisses. This set of premisses is divided in to three subsets, viz. the domain constraints, the structural facts and the remaining facts. The domain constraints are:

1. $on(x, y) \wedge y \neq z \rightarrow \neg on(x, z)$
2. $on(x, y) \wedge z \neq x \wedge y \neq floor \rightarrow \neg on(z, y)$
3. $rounded(x) \rightarrow \neg on(x, y)$
4. $duct(d) \wedge \exists x.on(x, d) \rightarrow blocked(d)$
5. $\exists x.on(x, table) \leftrightarrow obscured(picture)$

6. $blocked(duct1) \wedge blocked(duct2) \leftrightarrow stuffy(room)$

The structural facts are:

7. $rounded(bird)$
8. $rounded(plant)$
9. $duct(duct1)$
10. $duct(duct2)$
11. $in(bottom\_shelf, bookcase)$
12. $in(top\_shelf, bookcase)$

The situational facts are:

13. $on(bird, top\_shelf)$
14. $on(tv, bottom\_shelf)$
15. $on(chest, floor)$
16. $on(plant, duct2)$
17. $on(bookcase, floor)$
18. $blocked(duct2)$
19. $\neg obscured(picture)$
20. $\neg stuffy(room)$

The domain constraints are complemented with the *unique name assumption* (UNA).

Clearly, the situational facts are less reliable than the structural facts and the domain constraints. Furthermore, facts added by recent actions are on average more reliable than facts added by less recent actions.

Now suppose that we move the *plant* from *duct2* to the *table*. This can be described by adding the situational fact $on(plant, table)$. From the new set of premisses we can derive two inconsistencies;

$$\{\exists x. on(x, table) \leftrightarrow obscured(picture), \\ \neg obscured(picture), on(plant, table)\}$$

and

$$\{on(x, y) \wedge y \neq z \rightarrow \neg on(x, z), \\ on(plant, duct2), on(plant, table)\}.$$

The least reliable premisses in these sets of premisses are respectively the facts $\neg obscured(picture)$ and $on(plant, duct2)$. Hence, they have to be removed from the set of premisses.

12 Conclusions

In this paper a logic for reasoning with inconsistent knowledge has been described. One of the original motivations for developing this logic was based on the view that default reasoning is as a special case of reasoning with inconsistent knowledge. To describe defaults in this logic, such as Poole's framework for default reasoning, formulas containing free variables can be used. These formulas denote a set of ground instances. If we do not generate these ground instances, but, by using unification of terms containing free variables, we reason with formulas containing free variables, we can derive conclusions representing sets of instances. This would seem to be a very useful property.

Since, in the logic described here a default rule can only be described by using material implication, a default rule has a contraposition. It is possible, however, the contraposition may not hold for default rules. For example, the contraposition of the default rule: 'someone who owns a driving licence, may drive a car' is not valid. A better candidate for a default reasoning would be Reiter's Default logic [14] or Brewka's aproach [3].

Although it is likely that the logic is not suited for default reasoning, it is suited for reasoning with knowledge coming from different and not fully reliable knowledge sources. For this use of the logic, it seems plausible that the logic satisfies the properties of system **P**. As was shown in the examples described in Section 10, the reliability relation can be given a plausible probabilistic and ontological interpretations. Furthermore, the current belief set with respect to the inferences made can be determined efficiently. One important disadvantage is that, given a set of premisses containing many inconsistencies and insufficient knowledge about the relative reliability, the number of possible belief sets can grow exponentially in the number minimal inconsistencies detected.

Appendix

Theorem 12 *Soundness*

For each $i \geq 0$:

if $P \Rightarrow \varphi \in \mathcal{A}_i^{\prec'}$, then:

$$P \subseteq \Sigma \text{ and } P \models \varphi.$$

Proof By the soundness of propositional logic,

if $P \vdash \varphi$, then $P \models \varphi$.

Therefore, we only have to prove that for each $i \geq 0$:

if $P \Rightarrow \varphi \in \mathcal{A}_i^{\prec'}$, then $P \subseteq \Sigma$ and $P \vdash \varphi$.

We can prove this by induction on the index i of $\mathcal{A}_i^{\prec'}$.

- For $i = 0$:

  $\{\varphi\} \Rightarrow \varphi \in \mathcal{A}_0^{\prec'}$ if and only if $\varphi \in \Sigma$.

  Therefore, $\{\varphi\} \vdash \varphi$.

- Proceeding inductively, suppose that $P \Rightarrow \varphi \in \mathcal{A}_i^{\prec'}$.
  Then:

  $P \Rightarrow \varphi \in \mathcal{A}_i^{\prec'}$ if and only if $P \Rightarrow \varphi \in \mathcal{A}_{i-1}^{\prec'}$ or $P \Rightarrow \varphi$ has been added by Rule 1 or 2.

  - If $P \Rightarrow \varphi \in \mathcal{A}_{i-1}^{\prec'}$, then, by the induction hypothesis,

    $P \subseteq \Sigma$ and $P \vdash \varphi$.

  - If $P \Rightarrow \varphi$ is introduced by Rule 1, then it is an axiom.
    Therefore, $P = \varnothing$ and $\vdash \varphi$.
  - If $P \Rightarrow \varphi$ is introduced by Rule 2, then there is a $Q \Rightarrow \psi \in \mathcal{A}_{i-1}^{\prec'}$, $R \Rightarrow (\psi \rightarrow \varphi) \in \mathcal{A}_{i-1}^{\prec'}$.
    Therefore, $P = (Q \cup R)$.
    According to the induction hypothesis there holds:

    $Q, R \subseteq \Sigma$,

    $Q \vdash \psi$

    and

    $R \vdash \psi \rightarrow \varphi$.

    Hence:

    $P \subseteq \Sigma$ and $P \vdash \varphi$.

□

Theorem 13 *Completeness*
For each $P \subseteq \Sigma$:

if $P \models \varphi$, then there exists a $Q \subseteq P$ such that for some $i \geq 0$:

$$Q \Rightarrow \varphi \in \mathcal{A}_i^{\prec'}.$$

Proof Let $P \subseteq \Sigma$ and $P \models \varphi$.
By the completeness of propositional logic,

if $P \models \varphi$, then $P \vdash \varphi$.

Since $P \vdash \varphi$, there exists a deduction sequence $\langle \varphi_1, \varphi_2, ..., \varphi_n \rangle$ such that $\varphi_n = \varphi$ and for each $j < n$: either

- $\varphi_j \in P$, or
- φ_j is an axiom, or
- there exists a φ_k and a φ_l with $k, l < j$ and $\varphi_l = \varphi_k \to \varphi_j$.

The theorem will be proven, using induction on the length n of the deduction sequence.

- For $n = 1$, $\langle \varphi_1 \rangle$ is the deduction sequence for $P \vdash \varphi$.
  - If $\varphi_1 \in P$, then $\{\varphi_1\} \Rightarrow \varphi_1 \in \mathcal{A}_0^{\prec'}$.
  - If φ_1 is an axiom, then there exists some $i_0 \geq 1$ such that:
    $\mathcal{A}_{i_1}^{\prec'} = \mathcal{A}_{i_1-1}^{\prec'} \cup \{\varnothing \Rightarrow \varphi_1\}$ and $\varnothing \Rightarrow \varphi_1$ is added by Rule 1.

  Hence, the theorem holds for deduction sequences of length 1.

- Proceeding inductively, let $\langle \varphi_1, \varphi_2, ..., \varphi_n \rangle$ be a deduction sequence for $P \vdash \varphi_n$.
  - If $\varphi_n \in P$, then $\{\varphi_n\} \Rightarrow \varphi_n \in \mathcal{A}_0^{\prec'}$.
  - If φ_n is an axiom, then there exists an i_n such that:
    $\mathcal{A}_{i_n}^{\prec'} = \mathcal{A}_{i_n-1}^{\prec'} \cup \{\varnothing \Rightarrow \varphi_n\}$ and $\varnothing \Rightarrow \varphi_n$ is added by Rule 1.
  - If there exists a φ_k and a φ_l with $k, l < n$ and $\varphi_l = \varphi_k \to \varphi_n$, then, by the induction hypothesis, there exists some i_k and some i_l such that:

    $$R \Rightarrow \varphi_k \in \mathcal{A}_{i_k}^{\prec'},$$

    $$S \Rightarrow (\varphi_k \to \varphi_n) \in \mathcal{A}_{i_l}^{\prec'}$$

    and

    $$R, S \subseteq P.$$

  Because of the fairness Assumption 11, there must exist an i_n with $i_k, i_l < i_n$ such that:

$$\mathcal{A}_{i_n}^{\prec'} = \mathcal{A}_{i_n-1}^{\prec'} \cup \{R \cup S \Rightarrow \varphi_n\} \text{ and } R \cup S \Rightarrow \varphi_n \text{ is added by Rule 2.}$$

Hence there exists some i_n such that $Q \Rightarrow \varphi_n \in \mathcal{A}_{i_n}^{\prec'}$ and $Q \subseteq P$.

□

Theorem 14 *Soundness*
For each $i \geq 0$:

if $P \not\Rightarrow \varphi \in \mathcal{A}_i^{\prec'}$, then:

$$(P \cup \{\varphi\}) \subseteq \Sigma, \text{ and } (P \cup \{\varphi\}) \text{ is not satisfiable.}$$

Proof The theorem will be proven using induction to the index i of the set of arguments $\mathcal{A}_i^{\prec'}$.

- For $i = 0$ the theorem holds vacuously, because there is no $P \not\Rightarrow \varphi \in \mathcal{A}_0^{\prec'}$.
- Proceeding inductively, suppose that $P \not\Rightarrow \varphi \in \mathcal{A}_i^{\prec'}$.
  $P \not\Rightarrow \varphi \in \mathcal{A}_i^{\prec'}$ if and only if $P \not\Rightarrow \varphi \in \mathcal{A}_{i-1}^{\prec'}$ or $P \not\Rightarrow \varphi$ has been added by Rule 3.
  - If $P \not\Rightarrow \varphi \in \mathcal{A}_{i-1}^{\prec'}$, then, by the induction hypothesis, $(P \cup \{\varphi\}) \subseteq \Sigma$ and $(P \cup \{\varphi\})$ is not satisfiable.
  - If $P \not\Rightarrow \varphi$ is introduced by Rule 3, then there exists an $R \Rightarrow \psi \in \mathcal{A}_{i-1}^{\prec'}$ and a $Q \Rightarrow \neg\psi \in \mathcal{A}_{i-1}^{\prec'}$ such that:

    $$\varphi = \min_{\prec'}(Q \cup R) \text{ and } P = (R \cup Q)\backslash\varphi.$$

    By Theorem 12:

    $$R, Q \subseteq \Sigma,$$

    $$R \mid\!\!\sim \psi \text{ and } Q \mid\!\!\sim \neg\psi.$$

    Hence $(P \cup \{\varphi\}) \subseteq \Sigma$, and $(P \cup \{\varphi\})$ is inconsistent.
    Since inconsistency implies unsatisfiability:

    $$(P \cup \{\varphi\}) \subseteq \Sigma \text{ and } (P \cup \{\varphi\}) \text{ is not satisfiable.}$$

□

Theorem 15 *Completeness*
For each $P \subseteq \Sigma$:

if P is a minimal unsatisfiable set of premisses and $\varphi = \min_{\prec'}(P)$, then for some $i \geq 0$:

$$P\backslash\varphi \not\Rightarrow \varphi \in \mathcal{A}_i^{\prec'}.$$

Proof Let P be a minimal unsatisfiable subset of Σ with $\varphi = \min_{\prec'}(P)$.
Since P is a minimal unsatisfiable set, P is a minimal inconsistent set.
Therefore, there exists a proposition ψ such that:

$$P \vdash \psi \text{ and } P \vdash \neg\psi.$$

By Theorem 13 there exists a $j, k \geq 0$ such that:

$$S \Rightarrow \psi \in \mathcal{A}_j^{\prec'}, S \subseteq P$$

and

$$T \Rightarrow \neg\psi \in \mathcal{A}_k^{\prec'}, T \subseteq P.$$

Hence, $(S \cup T) \subseteq P$.
Since P is a minimal inconsistent set of premisses:

$$(S \cup T) = P.$$

Because of the fairness Assumption 11 there exists an $i > j, k$ such that:

$$(P\backslash\varphi) \not\Rightarrow \varphi \in \mathcal{A}_i^{\prec'}.$$

□

Property 18 For every i, the set $\Delta_i^{\prec'}$ exists and is unique.

Proof Existence Let $\delta_0 \supset \delta_1 \supset ... \supset \delta_k$ be a sequence of sets of premisses such that:

- $\Sigma = \delta_0$,
- $\delta_{j+1} = \delta_j \backslash \{\varphi\}$ where φ is the most reliable premiss in δ_j such that $P \not\Rightarrow \varphi \in \mathcal{A}_i^{\prec'}$ and $P \subseteq \delta_j$.

Then, by induction on the index of the sequence, we can prove that:

$$\Sigma \backslash Out_i^{\prec'}(\delta_j) \subseteq \delta_j.$$

- For $j = 0$, clearly, there holds $\Sigma \backslash Out_i^{\prec'}(\delta_0) \subseteq \delta_0$.
- Proceeding inductively, let the induction hypothesis hold for $\ell \leq j$.
  If $\Sigma \backslash Out_i^{\prec'}(\delta_j) \subset \delta_j$, then there exists a most reliable $\varphi \in \delta_j$ such that $P \not\Rightarrow \varphi$ and $P \subseteq \delta_j$.
  Now suppose that $\Sigma \backslash Out_i^{\prec'}(\delta_{j+1}) \not\subseteq \delta_{j+1}$.
  Then there exists a $\psi \notin Out_i^{\prec'}(\delta_{j+1})$ and $\psi \notin \delta_{j+1}$.
  Suppose that $\psi \in \delta_j$.
  Then $\psi = \varphi$.
  Since φ is the most reliable premiss such that $P \not\Rightarrow \varphi$ and $P \subseteq \delta_j$, $P \subseteq \delta_{j+1}$.
  Hence, $\psi \in Out_i^{\prec'}(\delta_{j+1})$.
  Contradiction.
  Hence, $\psi \notin \delta_j$ and, by the construction of δ_j, $\varphi \prec' \psi$.
  Since $\psi \notin \delta_j$, by the induction hypothesis, $\psi \in Out(\delta_j)$.
  Therefore, there exists a $Q \not\Rightarrow \psi \in \mathcal{A}_i^{\prec'}$ and $Q \subseteq \delta_j$.
  Since $\varphi \prec' \psi$, $Q \subseteq \delta_{j+1}$.
  Hence, $\psi \in Out_i^{\prec'}(\delta_{j+1})$.
  Contradiction.
  Hence, $\Sigma \backslash Out_i^{\prec'}(\delta_{j+1}) \subseteq \delta_{j+1}$.

Let k be the highest index in the sequence.
Then there does not exist a $\varphi \in \delta_k$ such that $P \not\Rightarrow \varphi \in \mathcal{A}_i^{\prec'}$ and $P \subseteq \delta_k$.
Hence, $\Sigma \backslash Out_i^{\prec'}(\delta_k) = \delta_k$, otherwise there would exist a $\varphi \in \delta_k$ such that $P \not\Rightarrow \varphi \in \mathcal{A}_i^{\prec'}$ and $P \subseteq \delta_k$.

Hence, there exists at least one $\Delta_i^{\prec'}$ such that:

$$\Delta_i^{\prec'} = \Sigma \backslash Out_i^{\prec'}(\Delta_i^{\prec'}).$$

Uniqueness Suppose $\Delta_i^{\prec'}$ is not unique.
Then there exist at least two different subsets $\Delta_i^{\prec'}, \Delta_i'^{\prec'} \subset \Sigma$ satisfying Definition 17.
Let φ be the most reliable proposition in Σ such that:

$$\varphi \notin \Delta_i^{\prec'} \text{ and } \varphi \in \Delta_i'^{\prec'}.$$

or

$$\varphi \notin \Delta_i'^{\prec'} \text{ and } \varphi \in \Delta_i^{\prec'}.$$

Let us consider the first case. Note that the second case is similar. Then, there exists a $P \not\Rightarrow \varphi \in \mathcal{A}_i^{\prec'}$.
By Theorem 15 there holds:

$P \cup \{\varphi\}$ is unsatisfiable.

Therefore, there exists a minimal inconsistent set of premisses Q with $\varphi = \min_{\prec'}(Q)$.
Since $\varphi \notin \Delta_i^{\prec'}$ and $\varphi \in \Delta_i'^{\prec'}$, there exists a $\psi \in Q$ such that:

$$\psi \in \Delta_i^{\prec'}, \psi \notin \Delta_i'^{\prec'} \text{ and } \varphi \prec \psi.$$

Hence, φ is not the most reliable proposition in Σ such that:

$$\varphi \notin \Delta_i^{\prec'} \text{ and } \varphi \in \Delta_i'^{\prec'}.$$

or

$$\varphi \notin \Delta_i'^{\prec'} \text{ and } \varphi \in \Delta_i^{\prec'}.$$

Contradiction.

Hence $\Delta_i^{\prec'}$ is unique.

□

Property 20 For each $\varphi \in B_i^{\prec'}$: $\Delta_i^{\prec'} \mathrel{|\!\!\sim} \varphi$.

Proof Suppose $\varphi \in B_i^{\prec'}$.
Then there exists a $P \Rightarrow \varphi \in \mathcal{A}_i^{\prec'}$ such that:

$$P \subseteq \Delta_i^{\prec'}.$$

Therefore, by Theorem 12:

$$P \mathrel{|\!\!\sim} \varphi \text{ and } P \subseteq \Delta_i^{\prec'}.$$

Hence, $\Delta_i^{\prec'} \mathrel{|\!\!\sim} \varphi$. □

Property 22 $\Delta_\infty^{\prec'}$ is maximal consistent.

Proof Suppose that $\Delta_\infty^{\prec'}$ is inconsistent.
Then there exists a minimal inconsistent subset M of $\Delta_\infty^{\prec'}$.
Let $\varphi = \min_{\prec'}(M)$.
Then by Theorem 15 there exists an i with

$$P \not\Rightarrow \varphi \in \mathcal{A}_i^{\prec'}$$

Hence $P \not\Rightarrow \varphi \in \mathcal{A}_\infty^{\prec'}$.
Because $P \subseteq \Delta_\infty^{\prec'}$, $\varphi \notin \Delta_\infty^{\prec'}$.
Contradiction.

Suppose that some $\Delta_\infty^{\prec'}$ is not maximal consistent.
Then there exists a $\varphi \in (\Sigma \backslash \Delta_\infty^{\prec'})$ and $\{\varphi\} \cup \Delta_\infty^{\prec'}$ is consistent.
Since $\varphi \in (\Sigma \backslash \Delta_\infty^{\prec'})$, $\varphi \in Out_\infty^{\prec}(\Delta_\infty^{\prec'})$.
Therefore, there exists a $P \not\Rightarrow \varphi \in \mathcal{A}_\infty^{\prec'}$ and $P \subseteq \Delta_\infty^{\prec'}$.
Since $P \not\Rightarrow \varphi \in \mathcal{A}_\infty^{\prec'}$, $P \cup \{\varphi\}$ is inconsistent.
Hence $\Delta_\infty^{\prec'} \cup \{\varphi\}$ is inconsistent.
Contradiction. □

Property 23

$$B_\infty = Th(\Delta_\infty^{\prec'})$$

where $Th(S) = \{\varphi \mid S \vdash \varphi\}$

Proof According to Property 20:

if $\varphi \in B_\infty$, then $\Delta_\infty^{\prec'} \vdash \varphi$.

Suppose there exists a φ such that:

$\varphi \notin B_\infty$ and $\varphi \in Th(\Delta_\infty^{\prec'})$.

Since $\varphi \in Th(\Delta_\infty^{\prec'})$, $\Delta_\infty^{\prec'} \vdash \varphi$.
By Theorem 13 there exists some i and some $P \Rightarrow \varphi \in \mathcal{A}_i^{\prec'}$ such that:

$P \subseteq \Delta_\infty^{\prec'}$.

Therefore, there exists a $P \Rightarrow \varphi \in \mathcal{A}_\infty^{\prec'}$ such that:

$P \subseteq \Delta_\infty^{\prec'}$.

Hence, by Definition 20:

$\varphi \in B_\infty$.

Contradiction.

Hence $B_\infty = Th(\Delta_\infty^{\prec'})$. □

Theorem 24 Let $\langle \Sigma, \prec \rangle$ be a reliability theory.

Then there holds:

$$\mathcal{R} = \{\Delta_\infty^{\prec'} \mid \text{ for some linear extension } \prec' \text{ of } \prec, \Delta_\infty^{\prec'} \text{ can be derived}\}.$$

Proof Let $\Delta_\infty^{\prec'}$ be a set of believed premisses given a linear extension $\prec'$ of $\prec$. Furthermore, let $\sigma_1, ..., \sigma_m$ be an enumeration of Σ such that for every $\sigma_j \prec' \sigma_k$: $k < j$.
Clearly, given this enumeration of Σ, $\Delta_\infty^{\prec'}$ will satisfy Definition 6.

Let D be a most reliable consistent set of premisses according to Definition 6 given an enumeration $\sigma_1, ..., \sigma_n$ of Σ. Furthermore, let $\prec'$ be a linear extension of $\prec$ such that for each $k < j$: $\sigma_j \prec' \sigma_k$.

Now suppose that:

$$D \neq \Sigma \backslash Out_\infty^{\prec'}(D).$$

Hence, there exists a most reliable premiss $\varphi \in \Sigma$ such that either

$$\varphi \in D \text{ and } \varphi \in Out_\infty^{\prec'}(D)$$

or

$$\varphi \notin D \text{ and } \varphi \notin Out_\infty^{\prec'}(D).$$

Suppose that $\varphi \in D$ and $\varphi \in Out_\infty^{\prec'}(D)$.
Since $\varphi \in Out_\infty^{\prec'}(D)$, for some $P \not\Rightarrow \varphi \in \mathcal{A}_\infty^{\prec'}$ there holds:

$$P \subseteq D.$$

Since $P \subseteq D$ and since $\varphi \in D$, D is inconsistent.
By Definition 6, however, D must be consistent.
Contradiction.

Suppose that $\varphi \notin D$ and $\varphi \notin Out_\infty^{\prec'}(D)$.
Since $\varphi \notin D$, there exists a minimal inconsistent of premisses $\{\sigma_{i_1}, ..., \sigma_{i_l}\}$ where i_j are indexes of the enumeration of Σ, $i_j < i_{j-1}$, $\{\sigma_{i_1}, ..., \sigma_{i_{l-1}}\} \subseteq D$ and $\varphi = \sigma_{i_l}$.
Therefore, by Theorem 15 an by the above given definition of $\prec'$:

$$\{\sigma_{i_1}, ..., \sigma_{i_{l-1}}\} \not\Rightarrow \sigma_{i_l} \in \mathcal{A}_\infty^{\prec'}$$

Hence, $\varphi \in Out_\infty^{\prec'}(D)$
Contradiction. □

Property 30 Let $\langle\Sigma, \prec\rangle$ be a reliability theory and let $\sqsubset$ be the preference relation over interpretations defined the reliability theory.

$\sqsubset$ is irreflexive and transitive.

Proof Suppose the $\sqsubset$ is not irreflexive.
Then for some interpretations $\mathcal{M}, \mathcal{N}$, $\mathcal{M} \sqsubset \mathcal{N}$ and $\mathcal{N} \sqsubset \mathcal{M}$.
Since $Prem(\mathcal{M}) \neq Prem(\mathcal{N})$, for some φ, $\varphi \in Prem(\mathcal{M}) \backslash Prem(\mathcal{N})$ or $\varphi \in Prem(\mathcal{N}) \backslash Prem(\mathcal{M})$.

Consider the former case.
Let φ be a most reliable premiss such that $\varphi \in Prem(\mathcal{M}) \backslash Prem(\mathcal{N})$. [4]
Since $\mathcal{M} \sqsubset \mathcal{N}$, there is a $\psi \in Prem(\mathcal{N}) \backslash Prem(\mathcal{M})$ such that $\varphi \prec \psi$.
Since $\psi \in Prem(\mathcal{N}) \backslash Prem(\mathcal{M})$ and $\mathcal{N} \sqsubset \mathcal{M}$, there is an $\eta \in Prem(\mathcal{M}) \backslash Prem(\mathcal{N})$ such that $\psi \prec \eta$.
Since $\varphi \prec \psi \prec \eta$, φ cannot be a most reliable premiss such that $\varphi \in Prem(\mathcal{M}) \backslash Prem(\mathcal{N})$.
Contradiction.

The latter case is similar and also results in a contradiction.

Hence, $\sqsubset$ is irreflexive.

Suppose that $\sqsubset$ is not transitive.
Then there exist structures $\mathcal{L}, \mathcal{M}, \mathcal{N}$ such that $\mathcal{L} \sqsubset \mathcal{M} \sqsubset \mathcal{N}$ but $\mathcal{L} \not\sqsubset \mathcal{N}$.
Therefore, for some $\varphi \in Prem(\mathcal{L}) \backslash Prem(\mathcal{N})$ there is no $\psi \in Prem(\mathcal{N}) \backslash Prem(\mathcal{L})$ such that $\varphi \prec \psi$. Let φ be the most reliable premiss for which the above holds.
The following cases can be distinguished:

- Suppose that $\varphi \in Prem(\mathcal{M})$.
  Then $\varphi \in Prem(\mathcal{M}) \backslash Prem(\mathcal{N})$ and therefore, there is an $\eta \in Prem(\mathcal{N}) \backslash Prem(\mathcal{M})$ such that $\varphi \prec \eta$.
  Let η be the most reliable premiss such that $\eta \in Prem(\mathcal{N}) \backslash Prem(\mathcal{M})$ and $\varphi \prec \eta$.

  Suppose $\eta \notin Prem(\mathcal{L})$.
  Then there is a $\psi \in Prem(\mathcal{N}) \backslash Prem(\mathcal{L})$, namely $\psi = \eta$, such that $\varphi \prec \psi$.
  Contradiction.

  Hence, $\eta \in Prem(\mathcal{L})$.
  Therefore, $\eta \in Prem(\mathcal{L}) \backslash Prem(\mathcal{M})$, implying that there is a $\mu \in Prem(\mathcal{M}) \backslash Prem(\mathcal{L})$ such that $\eta \prec \mu$.
  Suppose that $\mu \in Prem(\mathcal{N})$.
  Then there is a $\psi \in Prem(\mathcal{N}) \backslash Prem(\mathcal{L})$, namely $\psi = \mu$, such that $\varphi \prec \psi$ because $\varphi \prec \eta \prec \mu$ and $\prec$ is transitively closed.
  Contradiction.

[4] A most reliable premiss exist because Σ is finite and $\prec$ is defined over Σ.

Hence, $\mu \notin Prem(\mathcal{N})$.
Therefore, there is a $\xi \in Prem(\mathcal{N})\backslash Prem(\mathcal{M})$ such that $\mu \prec \xi$.
Since $\varphi \prec \eta \prec \mu \prec \xi$ and $\prec$ is transitive, η is not the most reliable premiss such that $\eta \in Prem(\mathcal{N})\backslash Prem(\mathcal{M})$ and $\varphi \prec \eta$.
Contradiction.

- Suppose that $\varphi \notin Prem(\mathcal{M})$.
Then there is a $\eta \in Prem(\mathcal{M})\backslash Prem(\mathcal{L})$ such that $\varphi \prec \eta$.
Let η be the most reliable premiss such that $\eta \in Prem(\mathcal{M})\backslash Prem(\mathcal{L})$ and $\varphi \prec \eta$.

  Suppose that $\eta \in Prem(\mathcal{N})$.
  Then there is a $\psi \in Prem(\mathcal{N})\backslash Prem(\mathcal{L})$, namely $\psi = \eta$, such that $\varphi \prec \psi$.
  Contradiction.
  Hence, $\eta \notin Prem(\mathcal{N})$.
  Therefore, there is a $\mu \in Prem(\mathcal{N})\backslash Prem(\mathcal{M})$ such that $\eta \prec \mu$.
  Suppose $\mu \notin Prem(\mathcal{L})$.
  Then there is a $\psi \in Prem(\mathcal{N})\backslash Prem(\mathcal{L})$, namely $\psi = \mu$, such that $\varphi \prec \psi$ because $\varphi \prec \eta \prec \mu$ and $\prec$ is transitively closed.
  Contradiction.
  Hence, $\mu \in Prem(\mathcal{L})$.
  Therefore, there is a $\xi \in Prem(\mathcal{M})\backslash Prem(\mathcal{L})$ such that $\mu \prec \xi$.
  Since $\varphi \prec \eta \prec \mu \prec \xi$ and $\prec$ is transitive, η is not the most reliable premiss such that $\eta \in Prem(\mathcal{M})\backslash Prem(\mathcal{L})$ and $\varphi \prec \eta$.
  Contradiction.

Hence, $\sqsubset$ is transitive. □

Theorem 32 Let $\langle \Sigma, \prec \rangle$ be a reliability theory. Furthermore, let $\mathcal{R}$ be the corresponding set of all most reliable consistent sets of premisses. Then:

$$Mod_{\sqsubset}(\langle \Sigma, \prec \rangle) = \bigcup_{\Delta_\infty^{\prec'} \in \mathcal{R}} Mod(\Delta_\infty^{\prec'})$$

where $Mod(S)$ denotes the set of classical models for a set of propositions S.

Proof The proof of

$$Mod_{\sqsubset}(\langle \Sigma, \prec \rangle) = \bigcup_{\Delta_\infty^{\prec'} \in \mathcal{R}} Mod(\Delta_\infty^{\prec'})$$

can be divided into the proof of the soundness

$$Mod_{\sqsubset}(\langle \Sigma, \prec \rangle) \subseteq \bigcup_{\Delta_\infty^{\prec'} \in \mathcal{R}} Mod(\Delta_\infty^{\prec'})$$

and the proof of the completeness

$$\bigcup_{\Delta_\infty^{\prec'} \in \mathcal{R}} Mod(\Delta_\infty^{\prec'}) \subseteq Mod_{\sqsubset}(\langle \Sigma, \prec \rangle)$$

of the logic.

Completeness Suppose that for some $\Delta_\infty^{\prec'} \in \mathcal{R}$ and some $\mathcal{M} \in Mod(\Delta_\infty^{\prec'})$:

$$\mathcal{M} \notin Mod_{\sqsubset}(\langle \Sigma, \prec \rangle).$$

Then there exists a structure $\mathcal{N}$:

$$\mathcal{M} \sqsubset \mathcal{N}.$$

$Prem(\mathcal{M}) = \Delta_\infty^{\prec'}$ because $\Delta_\infty^{\prec'}$ is a maximal consistent set of premisses.
Therefore, according to Proposition 22:

$$\Delta_\infty^{\prec'} \not\subset Prem(\mathcal{N}).$$

Let $\varphi \in \Delta_\infty^{\prec'}$ be the *most reliable* premiss according to the linear extension $\prec'$ of $\prec$, such that $\varphi \in (\Delta_\infty^{\prec'} \backslash Prem(\mathcal{N}))$.
Now by Definition 29 there exists a $\psi \in (Prem(\mathcal{N}) \backslash \Delta_\infty^{\prec'})$ such that $\varphi \prec \psi$.
Since $\psi \notin \Delta_\infty^{\prec'}$, there exists a $P \not\Rightarrow \psi \in \mathcal{A}_\infty^{\prec'}$ such that:

$$P \subseteq \Delta_\infty^{\prec'}.$$

Now, $P \not\subseteq Prem(\mathcal{N})$, otherwise $Prem(\mathcal{N})$ would be inconsistent.
Hence, there exists a $\mu \in P$:

$$\mu \in (\Delta_\infty^{\prec'} \backslash Prem(\mathcal{N})).$$

Since $P \not\Rightarrow \psi \in \mathcal{A}_\infty^{\prec'}$, $\psi \prec' \mu$.
Hence, $\varphi \prec' \psi \prec' \mu$.
Contradiction.

Hence,

$$\bigcup_{\Delta_\infty^{\prec'} \in \mathcal{R}} Mod(\Delta_\infty^{\prec'}) \subseteq Mod_{\sqsubset}(\langle \Sigma, \prec \rangle).$$

Soundness Let $\mathcal{M} \in Mod_{\sqsubset}(\langle \Sigma, \prec \rangle)$.

Then, $\mathcal{M} \in \bigcup_{\Delta_\infty^{\prec'} \in \mathcal{R}} Mod(\Delta_\infty^{\prec'})$ has to be proven.

Note that $Prem(\mathcal{M})$ is a maximal consistent set of premisses because otherwise, there would exists an $\mathcal{N}$ such that $\mathcal{M} \sqsubset \mathcal{N}$.

Hence, for some linear extension $\prec'$ of $\prec$, $Prem(\mathcal{M}) = \Delta_\infty^{\prec'}$ has to be proven.

The proof is based on constructing a linear extension $\prec'$ of $\prec$.

Staring from the most reliable premiss in $Prem(\mathcal{M})$ given the reliability relation $\prec$, and an initial reliability relation $\prec_0^* = \prec$, an extension $\prec_{|Prem(\mathcal{M})|}^*$ of $\prec$ is constructed.

Let $\varphi \in Prem(\mathcal{M})$ be a most reliable premiss given $\prec$ that has not been addressed yet, and let $\prec_i^*$ be the reliability relation constructed so far.

Create $\prec_{i+1}^*$ by adding $\eta \prec \varphi$ to $\prec_i^*$ for every minimal inconsistent subset P of Σ such that $\varphi \in P$, and for every $\eta \in P \backslash \varphi$ such that $\varphi \not\prec_i^* \eta$, and subsequently taking the transitive closure.

The correctness of the of the reliability relation $\prec_{i+1}^*$ depends on the condition $\varphi \not\prec_i^* \eta$. There are three cases:

- Suppose $\eta \in Prem(\mathcal{M})$ and that η is considered before φ.
  Then clearly, $\varphi \prec_i^* \eta$. Moreover, there must be a $\psi \in P$, $\psi \notin Prem(\mathcal{M})$ and the construction of $\prec^*$ guarantees that $\psi \prec_{i+1}^* \varphi$.
  Hence, the construction is correct.
- Suppose $\eta \in Prem(\mathcal{M})$ and that η is not considered before φ.
  Then, $\varphi \not\prec \eta$, and therefore, $\varphi \not\prec_i^* \eta$.
  Hence, the construction is correct.
- Suppose $\eta \notin Prem(\mathcal{M})$.
  Then there is a minimal inconsistent subset Q of Σ such that η is a least preferred premiss Q given $\prec$, and $Q \backslash \eta \subseteq Prem(\mathcal{M})$.
  Hence, the construction is correct.

After constructing a reliability relation $\prec_{|Prem(\mathcal{M})|}^*$, the final step is to take a linear extension of $\prec_{|Prem(\mathcal{M})|}^*$ in order to get $\prec'$.

□

Lemma 33 Let $\langle \Sigma, \prec \rangle$ be a reliability theory. Furthermore, let $\widehat{\alpha} = \{\mathcal{M} \mid \mathcal{M} \models \alpha\}$, let $\Sigma' = \Sigma \cup \{\alpha\}$ and let $\prec' = (\prec \cap (\Sigma/\alpha \times \Sigma/\alpha)) \cup \{\langle \varphi, \alpha \rangle \mid \varphi \in \Sigma/\alpha\}$.

Then $\mathcal{M} \in Mod_{\sqsubset'}(\langle \Sigma', \prec' \rangle)$ if and only if $\mathcal{M} \in \widehat{\alpha}$ and for no $\mathcal{N} \in \widehat{\alpha}$:

$$\mathcal{M} \sqsubset \mathcal{N}.$$

Proof The results presented in the following two items, will be used in the proof.

- Suppose that $\mathcal{M} \in \widehat{\alpha}$ and $\mathcal{N} \notin \widehat{\alpha}$, i.e. $\mathcal{M} \models \alpha$ and $\mathcal{N} \not\models \alpha$.
  Then by Definition 28:

  $$Prem(\mathcal{M}) \neq Prem(\mathcal{N}).$$

  Therefore,

  $$\alpha \in (Prem(\mathcal{M}) \backslash Prem(\mathcal{N}))$$

  and for each $\varphi \in (Prem(\mathcal{N}) \backslash Prem(\mathcal{M}))$ there holds:

  $$\varphi \prec' \alpha.$$

  Hence by Definition 29 for each $\mathcal{M} \in \widehat{\alpha}$ and $\mathcal{N} \notin \widehat{\alpha}$:

  $$\mathcal{N} \sqsubset' \mathcal{M}.$$

- Suppose that $\mathcal{M}, \mathcal{N} \in \widehat{\alpha}$.
  Since $\mathcal{M}, \mathcal{N} \models \alpha$, for each $\varphi \in (Prem(\mathcal{M}) \backslash Prem(\mathcal{N}))$ and for each $\psi \in (Prem(\mathcal{N}) \backslash Prem(\mathcal{M}))$:

  $$\varphi \prec \psi \text{ if and only if } \varphi \prec' \psi.$$

  Hence, for each $\mathcal{M}, \mathcal{N} \in \widehat{\alpha}$:

  $$\mathcal{N} \sqsubset' \mathcal{M} \text{ if and only if } \mathcal{N} \sqsubset \mathcal{M}.$$

Let $\mathcal{M} \in Mod_{\sqsubset'}(\langle \Sigma', \prec' \rangle)$.
The first item above shows that $\mathcal{M} \in \widehat{\alpha}$ and the second item shows that for no $\mathcal{N} \in \widehat{\alpha}$: $\mathcal{M} \sqsubset \mathcal{N}$.

Let $\mathcal{M} \in \widehat{\alpha}$ and for no $\mathcal{N} \in \widehat{\alpha}$: $\mathcal{M} \sqsubset \mathcal{N}$.
The first item shows that for no $\mathcal{N} \notin \widehat{\alpha}$: $\mathcal{M} \sqsubset' \mathcal{N}$.
The second item shows that $\mathcal{M}$ is preferred in $\widehat{\alpha}$ given $\sqsubset'$. □

Theorem 34 Let $\langle \Sigma, \prec \rangle$ be a reliability theory. Moreover, let $\langle S, l, < \rangle$ be a triple where the set of states S is the set of all possible interpretations for the language L, where $l : S \to S$ is the identity function, and where for each $\mathcal{M}, \mathcal{N} \in S$:

$$\mathcal{M} < \mathcal{N} \text{ if and only if } \mathcal{N} \sqsubset \mathcal{M}.$$

Then $\langle S, l, < \rangle$ is a *preferential model* [12].

Proof Since S is the set of all interpretations and since l is the identity function, a state corresponds one to one to an interpretation.
Therefore, since the relation $\sqsubset$ is an irreflexive and transitive partial order on interpretations, so is $<$ on S.

A transitive relation $<$ implies smoothness if S is finite.
In case of first order logic, there might exist an infinite long chain of preference, and therefore smoothness has to be proven.

Suppose that $<$ is not smooth.
Then by Lemma 33 for some proposition α and some $\mathcal{M} \in \widehat{\alpha}$ there holds neither that:

$$\mathcal{M} \in \mathit{Mod}_{\sqsubset'}(\langle \Sigma', \prec' \rangle),$$

nor does there exist a $\mathcal{N} \in \mathit{Mod}_{\sqsubset'}(\langle \Sigma', \prec' \rangle)$ such that:

$$\mathcal{M} \sqsubset \mathcal{N}.$$

If $\mathcal{M} \notin \mathit{Mod}_{\sqsubset'}(\langle \Sigma', \prec' \rangle)$, there must exists an $\mathcal{L}_1$: $\mathcal{M} \sqsubset \mathcal{L}_1$.
Suppose that for some $\mathcal{L}_i$ with $i \geq 1$ there does not exist an $\mathcal{L}_{i+1}$ such that:

$$\mathcal{L}_i \sqsubset \mathcal{L}_{i+1}.$$

Then $\mathcal{L}_i \in \mathit{Mod}_{\sqsubset'}(\langle \Sigma', \prec' \rangle)$.
According to Property 30,

$$\mathcal{M} \sqsubset \mathcal{L}_{i+1}.$$

Contradiction.

Hence, there exists an infinite sequence $\mathcal{M} \sqsubset \mathcal{L}_1 \sqsubset \mathcal{L}_2 \sqsubset ....$

For each $\mathcal{L}_i$ there exists a $\mathit{Prem}(\mathcal{L}_i) \subseteq \Sigma$.
Suppose that for some $j < i$: $\mathit{Prem}(\mathcal{L}_i) = \mathit{Prem}(\mathcal{L}_j)$.
Then $\mathcal{L}_j \not\sqsubset \mathcal{L}_i$.
Since $\sqsubset$ is transitive, $\mathcal{L}_j \sqsubset \mathcal{L}_i$.
Contradiction.

Hence, for each $\mathcal{L}_i, \mathcal{L}_j$ with $i \neq j$: $\mathit{Prem}(\mathcal{L}_i) \neq \mathit{Prem}(\mathcal{L}_j)$.
Since Σ is finite, for some i, j with $i \neq j$: $\mathit{Prem}(\mathcal{L}_i) = \mathit{Prem}(\mathcal{L}_j)$.
Contradiction.

Hence, $<$ is smooth.

Hence, $\langle S, l, < \rangle$ is a preferential model according to the definition of Kraus et al. $\square$

Theorem 35 Let $W = \langle S, l, < \rangle$ be a preferential model for $\langle \Sigma, \prec \rangle$. Then the following equivalence holds:

$\alpha \mathrel{|\!\sim}_W \beta$ if and only if

$\Sigma' = \Sigma \cup \{\alpha\}$,

$\prec' = (\prec \cap (\Sigma/\alpha \times \Sigma/\alpha)) \cup \{\langle \varphi, \alpha \rangle \mid \varphi \in \Sigma/\alpha\}$

and $\beta \in Th(\langle \Sigma', \prec' \rangle)$.

Proof According to Theorem 32:

$\beta \in Th(\langle \Sigma', \prec' \rangle)$ if and only if for each $\mathcal{M} \in Mod_{\sqsubset'}(\langle \Sigma', \prec' \rangle)$:

$\mathcal{M} \models \beta$.

Therefore, by Lemma 33:

$\beta \in Th(\langle \Sigma', \prec' \rangle)$ if and only if for each $\mathcal{M} \in \max_{\sqsubset}(\widehat{\alpha})$:

$\mathcal{M} \models \beta$.

Hence, by the definition of the non-monotonic entailment relation $\mathrel{|\!\sim}$ we have:

$\beta \in Th(\langle \Sigma', \prec' \rangle)$ if and only if $\alpha \mathrel{|\!\sim}_W \beta$.

$\square$

Theorem 37 Let belief set $K = Th(\langle \Sigma, \prec \rangle)$ be the set of theorems of the reliability theory $\langle \Sigma, \prec \rangle$.
Suppose that $K^*[\alpha]$ is the belief set of the premisses $\Sigma \cup \{\alpha\}$ with reliability relation:

$\prec' = (\prec \cap (\Sigma/\alpha \times \Sigma/\alpha)) \cup \{\langle \varphi, \alpha \rangle \mid \varphi \in \Sigma/\alpha\}$;

i.e. $K^*[\alpha] = \{\beta \mid \alpha \mathrel{|\!\sim}_W \beta\}$ where W is a preferential model for $\langle \Sigma, \prec \rangle$.
Then the following postulates are satisfied.

1. $K^*[\alpha]$ is a belief set.
2. $\alpha \in K^*[\alpha]$.
6. If $\vdash \alpha \leftrightarrow \beta$, then $K^*[\alpha] = K^*[\beta]$.

Proof

1. This follows from Property 23
2. Since $\alpha \mathrel{|\!\sim}_W \alpha$ (reflexivity), $\alpha \in K^*[\alpha]$.
6. Since $\dfrac{\models \alpha \leftrightarrow \beta, \alpha \mathrel{|\!\sim}_W \gamma}{\beta \mathrel{|\!\sim}_W \gamma}$ (left logical equivalence), if $\vdash \alpha \leftrightarrow \beta$, then $K^*[\alpha] = K^*[\beta]$.

□

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